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DIFFERENCE FREQUENCY HARMONIC ION HEATING USING WHISTLER MODES

by

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ABSTRACT

The heating of ions in a magnetized plasma by the use of the second order fields generated by the nonlinear mixing of two whistler modes is examined. The resulting kinetic equations describing the mixing and heating process are solved using the method of orbit integrations. Two techniques are available for the optimization of the energy dissipated by the ions. One is to allow the mixed wave to approach a natural mode in the plasma, resulting in a field resonance. A second method is to have the ions absorb energy through cyclotron damping. The resulting sensitivities to fluctuations in frequency, density, static magnetic field and inclination of the incident sources are given for the two techniques.

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LIST OF SYMBOLS

$\langle\ \rangle$	Average over v_x, v_y and v_z
$\langle\ \rangle_\perp$	Average over v_x and v_y
I_m	Imaginary part of
Re	Real part of
ε_x, ε_y, ε_z	Unit vectors, cartesian coordinates
$\underline{B}$	Magnetic flux density, gauss
c	Velocity of light
e	Electron charge
$\underline{E}$	Electric field intensity, statvolts/cm
$i = \sqrt{-1}$	
$\underline{J}$	Current density
k	Boltzmann's constant = 1.38×10^{-16} erg/deg (usually in the expression kT)
$\underline{K}$	Wave-propagation vector
n_+	Ion number density
n_-	Electron number density
$\underline{r}$	Position vector
$\underline{r}'$, $\underline{r}''$	Lagrangian space coordinates
t	Time
t'	Lagrangian time coordinate
$T_\parallel$	Temperature in the direction parallel to d-c magnetic field
$T_\perp$	Temperature in the direction perpendicular to d-c magnetic field

T^+, T^-	Ion and electron temperatures respectively
$\underline{v}$	Macroscopic particle velocity
$\underline{v}'$	Lagrangian coordinate
α_n	Parameter
$\beta = 8\pi nkT_{\parallel}/B_o^2$	Ion Kinetic pressure parallel to B_o ÷ magnetic pressure
$\gamma = \frac{K_x}{2\Omega} \frac{kT}{m}$	
$\lambda = \frac{K_x^2 \, kT}{\Omega^2 \, m}$	
ν_{ei}	Electron-ion collision frequency
ω_{p-}	Electron plasma frequency
ω_{p+}	Ion plasma frequency
ω_1, ω_2	Angular frequencies for first and second incident waves respectively
ω	Angular frequency for difference frequency wave
Ω_+	Ion gyration frequency
Ω_-	Electron gyration frequency
$\underline{\underline{M}}$	Dimensionless mobility tensor

A vector, v, is denoted by $\underline{v}$

A matrix, M, is denoted by $\underline{\underline{M}}$

$\underline{J}^*$	Complex conjugate of the vector $\underline{J}$

CHAPTER 1 INTRODUCTION

The present state of research indicates that the most promising heating techniques for the attainment of controlled thermonuclear conditions are those for which the energy is selectively coupled to the ions. Techniques dependent upon ion-electron collisional effects become inefficient for plasmas with temperatures greater than 100ev. For this reason, heating schemes which employ an ion cyclotron resonance effect are of great interest.

In this chapter, a summary of the techniques that have been used to date will be given, followed by a discussion on the method of difference frequency harmonic ion heating.

1.1 Heating Techniques Used to Date

Some of the schemes that have been employed to date in the heating of plasmas are enumerated below. Those involving magnetic pumping have been described by Berger[1]. A survey of experiments in which an ion cyclotron resonance effect is employed for ion heating is given by Hooke and Rothman[2].

(i) Ohmic Heating. The use of ohmic heating in the stellarator experiments is described by Rose and Clark[3] (pp. 440 to 449). Plasma heating arises strictly from the resistivity of a plasma to a current

which is produced by a unidirectional electric field pulse. This method of heating will be efficient only at low temperatures, because the plasma resistivity is proportional to $T^{-3/2}$, where T is the temperature of the plasma. Electrons are heated selectively in Ohmic heating, and since the transfer of electron energy to the ions is inefficient, this technique is not well suited for the heating of ions.

(ii) Collisional Heating. This method of heating is often referred to as magnetic pumping. The magnetic field is modulated in time, with the resulting heating in the perpendicular direction being transferred through collisions to the parallel direction. The principle involved is that the energy increase in a two dimensional or radial adiabatic compression is greater than in a three dimensional compression[3]. This technique is limited to low frequencies, because at high frequencies only the surface layers are compressed, resulting in only the surface layers being heated.

(iii) Transit Time Heating. In this method of ion heating, the magnetic field is modulated spatially and in time. This process will heat the ions in a direction parallel to the magnetic field, with the conversion to random energy resulting through collisions. The basis of the method is that the magnetic moment of a particle may be considered to be constant if the frequency at which the magnetic field is modulated is sufficiently low.

(iv) Rapid Compression. Using a pulsed magnetic field, the rapid radial compression of a plasma can give ion energies of 400 (+) ev. However the conditions required for a positive energy yield in a thermonuclear reaction using a "fast-pinch" technique

implies an instantaneous release of approximately 10^{10} joules[4], which is equivalent to several tons of TNT.

(v) ICRH. A technique that has been used extensively for the heating of ions in a plasma is ion cyclotron resonance heating (ICRH). This technique has been used for ion heating in the B-65 stellerator[5], the C stellerator[6], the B-66 mirror device[7], and in various other mirror machines[8,9]. In ICRH, the energy in an ion cyclotron wave is converted into the random motion of ions through the process of cyclotron damping. Because the ions are accelerated directly, ICRH schemes are promising for the attainment of high ion temperatures.

In the schemes that have been employed to date, an ion cyclotron wave is generated in a region of the plasma where Ω_+ is typically in the range of .5 to .85. This wave is then allowed to propagate along a direction of decreasing magnetic field strength, until the local cyclotron frequency approaches that of the propagating wave. The wave energy will be absorbed by the ions in this region through cyclotron damping. This region of decreasing magnetic field is referred to as a "magnetic beach" because of the analogy to the breaking of water waves at a shoreline. In experiments of this nature, the electron temperature remains much less than the ion temperature because of the effects of electron radiation and the selective coupling of energy to the ions at the magnetic beach.

Three techniques that have been used for coupling the rf power from an external source to an ion cyclotron wave in the plasma are:

(1) Reverse-turn induction coil. This type of coil is normally referred to as a Stix coil. A complete analysis of its use in

coupling rf energy from an external source to a plasma is given by Stix[10] (Chapter 5). The Stix coil has been used for the generation of ion cyclotron waves in the B-65 and C-stellerators, and by the Kharkov group for a linear discharge in a uniform field[11]. The efficiency with which rf energy may be coupled to a plasma is in the range of 0.3 to 0.6 for plasmas with densities of $10^{12}/cm^3$ to $10^{13}/cm^3$. Powers up to 1MW at 25 M Hertz have been used[6].

An example of the results that may be obtained for the heating of plasmas by the use of a Stix coil are those given by Nazarov[12]. For a plasma with a radius of 3.5 cm, and a density of $10^{13}/cm^3$, an ion energy of 2 Kev was obtained. The electrons remained much cooler with a temperature of approximatley 30 ev.

The reverse coil is suited to the generation of ion cyclotron waves in plasmas of moderate density only, that is for densities of less than $10^{14}/cm^3$. It cannot be used for the efficient generation of the short wavelengths required for high density plasmas. That is, as the spacing between the sections of the coil become smaller than the coil diameter, the electromagnetic field of the coil decreases very rapidly as one moves radially inward toward the plasma column. Also, the Kharkov group[2] has reported that the use of a Stix coil at power levels greater than 10 to 20 KW has resulted in the density of the plasma rapidly decaying during the heating pulse. This would appear to place an upper bound on the maximum power that may be used.

(2) Coaxial Electrodes. Rf power may be applied to coaxial electrodes at the end of a mirror device for the generation of ion cyclotron waves in a plasma[13]. The theoretical coupling efficiency

may be estimated by considering the device as a coaxial waveguide, filled with a material with a dielectric constant of $4\pi c^2/B^2$. This approximation is valid for frequencies much lower than the ion cyclotron frequency.

In the experiment performed by Boley, Wilcox et al[14], the transfer of energy from a 1 MW, 8.3 M hertz oscillator to a plasma with a density of $6 \times 10^{12}/cm^3$ was found to be 65% efficient. Better than 90% of the wave energy was absorbed by the ions at the magnetic beach.

A similar experiment was performed by Shvets[9,15] on the Vikhr' (Whirlwind) device. A dense plasma was produced in a metal discharge tube. Ion cyclotron waves were then generated near the magnetic mirror, and propagated in the axial direction. The wave energy was absorbed by the ions through cyclotron damping at a magnetic beach. In this experiment, an ion temperature of 250 ev was obtained in a plasma of density of $2 \times 10^{14}/cm^3$. Some of the features of the device used by Shvets[9,15] for ion heating are:

(i) The system has a low input impedance and does not require high voltages for the introduction of rf power into the plasma.

(ii) Good Coupling between the plasma and the electrodes is obtained.

(iii) At the magnetic beach, the energy from the wave is initially coupled to ion motion which is perpendicular to the static magnetic field, thus improving stability.

(iv) Two gases may be heated simultaneously. In this particular case, argon ions reached the same temperature as the protons.

The use of a coaxial electrode scheme for the generation of ion cyclotron waves in a plasma is limited to mirror confining structures.

For instance, this scheme cannot be used for ion heating in a closed system.

(3) Spatially Rotating Magnetic Field. In the experiment of Karr, Knapp, and Risenfeld [8], a plasma was injected into a spatially rotating static magnetic field from a coaxial plasma gun. The transverse component of the magnetic field ($\sim$ 190 gauss) was produced by a helical winding placed coaxially in the mirror device. This field was superimposed on an axial field of approximately 3000 gauss. By relating the coil wavelength (λ_z) to the velocity of the plasma jet (v_z) by

$$\lambda_z = \frac{2\pi v_z}{\Omega_+}$$

the plasma saw, in its frame of reference, a rotating electromagnetic field at the ion cyclotron frequency. Through the ion cyclotron damping of this wave, approximately one half of the longitudinal energy in the jet was converted into transverse energy, producing a plasma with $T = 1$ Kev and a density of $10^{14}/cm^3$.

1.2 Difference Frequency Harmonic Ion Heating

In an attempt to overcome some of the problems associated with the heating schemes discussed in Section (1.1), James and Thompson[16] have suggested the use of the difference frequency harmonic which is generated by the nonlinear mixing of two waves, for ion heating. This

indirect method for generating low frequency waves ($\omega \sim \Omega_+$) in a plasma is inherently inefficient because the waves used in the nonlinear mixing process are selected such that they readily penetrate the plasma. However, since the difference frequency wave is generated internally in the plasma, a large volume of the plasma may be heated simultaneously. This may be contrasted with the absorption of the wave energy by the ions at a magnetic beach for the case where either a Stix coil or coaxial electrodes are used to couple rf energy to a plasma. By choosing the frequencies for the waves used in the nonlinear mixing process to be greater than the electron plasma and electron cyclotron frequencies (see Section (1.2.1)), ions may be heated in plasmas confined by either open or closed systems. This may be contrasted with the use of coaxial electrodes or a spatially rotating magnetic field for ion heating, in which case a mirror device (open system) is required.

In this thesis, the use of Whistlers in the nonlinear mixing process will be investigated. It will be shown that plasmas with densities in the range 10^{12} to $10^{16}/cm^3$ may be heated through the nonlinear mixing of whistler modes below the infrared region.

As mentioned above, the waves used in the nonlinear mixing process are selected such that they readily penetrate the plasma. Two possible choices for these waves are listed below.

(i) High Frequency Waves. These are waves with a frequency greater than or equal to the electron plasma and electron cyclotron frequencies. This category includes the use of lasers in the mixing process.

(ii) Whistler Waves. These are waves with a frequency in the range between the ion and electron cyclotron frequencies, and which propagate approximately parallel to the static magnetic field.

The power absorbed by the ions from the difference frequency wave will be shown to be proportional to the fourth power of the driven electric field intensity. Therefore, the successful implementation of difference frequency harmonic ion heating will require the use of driving waves with high electric field intensities.

The results for the nonlinear mixing of high frequency waves and that of whistlers are summarized in Sections (1.2.1) and (1.2.2) respectively.

1.2.1 H F Heating

Two schemes have been suggested recently[17,18] for the use of waves with frequencies greater than or equal to the electron plasma and cyclotron frequencies in the mixing process. In both schemes, the energy absorbed by the ions through the collisional damping of the mixed wave is optimized by a resonance in the magnitude of the second order fields. This is achieved by allowing the mixed wave to approach a natural mode in the plasma. In the above cases, this was chosen to be an extra-ordinary Alfvén wave propagating perpendicularly to the static magnetic field. The requirements for a field resonance place stringent restrictions on the allowable fluctuatuions in the

angle of incidence for the sources, the density of the plasma, the static magnetic field, and the frequency of the waves.

1.2.2 Whistler Heating

The nonlinear mixing of whistlers for the heating of ions in a magnetized plasma will be investigated in this thesis. In order that collisionless damping effects be included, a kinetic analysis will be used to describe the heating and mixing process.

The use of a second order ion current resonance, obtained by allowing the difference frequency wave to approach the ion cyclotron frequency for optimizing the power absorbed by the ions from the difference frequency wave will be investigated. It will be shown that this technique may be used to gain an order of magnitude in the power absorbed by the ions over that obtained by James and Thompson[17], as well as the relaxation of the restrictions on the allowable density fluctuations and perturbations in the angle of incidence for the incident waves. A gain of two orders of magnitude is obtained in the sensitivity to frequency disturbances, and one order of magnitude in the sensitivity to static magnetic field fluctuations.

The use of the second order ion current resonance coupled with the field resonance effect described in Section (1.2.1) will also be investigated. It will be shown that this technique may be used to gain two orders of magnitude in the power absorbed by the ions over

that obtained by James and Thompson[17]. A gain of two orders of magnitude is obtained in the sensitivity to frequency, density and d-c magnetic field fluctuations. The heating process is less sensitive by two orders of magnitude to perturbations in the angle of incidence for the incident waves.

CHAPTER 2 NONLINEAR INTERACTION OF WAVES IN A PLASMA

In this chapter, a kinetic analysis will be used to describe the nonlinear mixing of two waves. The waves will be assumed to have frequencies in the whistler frequency domain. Ion and electron contributions to the net current and charge perturbations will be considered. The selection of the static magnetic field will be such that the collisionless damping effects on the incident waves are negligible. A cold plasma analysis may be used to show that terms of order of the electron collision frequency divided by the electron gyro-frequency (ν_{ei}/Ω_-) are introduced by the collisional effects on the incident waves, provided the incident frequencies are not too close to a gyro resonant frequency. For typical plasmas under consideration, this term is of the order of 10^{-5}. In the following analysis, the collisional damping effects on the incident waves will be neglected. Collisional effects will be included in the second order induced currents.

2.1 Kinetic Equations

The kinetic approach used to describe the mixing and heating process begins with the collisionless Boltzmann equation expressed in Lagrangian coordinates. That is, the change in the distribution function with time is written in a coordinate frame

moving with the zero'th order trajectory of a particle. To solve the resulting kinetic equation, the method of orbit integration given in Stix[10], will be extended to include second order fields and currents (see Appendix A). Later in Section (2.4), the corrections accounting for effects of collisions between the species will be discussed.

The Boltzmann equation for the j'th species may be expressed as follows:

$$\frac{\partial f^j}{\partial t} + \underline{v} \cdot \frac{\partial f^j}{\partial \underline{r}} + \frac{Z_j e \varepsilon_j}{m_j}\left(\underline{E} + \frac{\underline{v} \times \underline{B}}{c}\right) \cdot \frac{\partial f^j}{\partial \underline{v}} = 0 \qquad \text{...(2.1)}$$

where $Z_j e$ = magnitude of the charge on species j

ε_j = sign of the charge on species j

m_j = mass of species j

The change in the distribution function along the zero'th order trajectory of a particle is given by

$$\left(\frac{df^j}{dt}\right)_0 = \frac{\partial f^j}{\partial t} + \frac{\partial f^j}{\partial \underline{r}} \cdot \frac{d\underline{r}}{dt} + \frac{\partial f^j}{\partial \underline{v}} \cdot \frac{d\underline{v}}{dt}$$

$$= \frac{\partial f^j}{\partial t} + \underline{v} \cdot \frac{\partial f^j}{\partial \underline{r}} + \frac{Z_j e \varepsilon_j}{m_j} \frac{\underline{v} \times \underline{B}_0}{c} \cdot \frac{\partial f^j}{\partial \underline{v}} \qquad \text{...(2.2)}$$

The solution to equation (2.2) will be expanded as follows:

$$\begin{bmatrix} f^+ \\ f^- \\ \underline{E} \\ \underline{B} \end{bmatrix} = \begin{bmatrix} f_0^+ \\ f_0^- \\ 0 \\ \underline{B}_0 \end{bmatrix} + \begin{bmatrix} f_1^+ \\ f_1^- \\ \underline{E}_1 \\ \underline{B}_1 \end{bmatrix} e^{i(\underline{K}_1\cdot\underline{r}-\omega_1 t)} + \begin{bmatrix} f_2^+ \\ f_2^- \\ \underline{E}_2 \\ \underline{B}_2 \end{bmatrix} e^{i(\underline{K}_2\cdot\underline{r}-\omega_2 t)} + \begin{bmatrix} f_3^+ \\ f_3^- \\ \underline{E}_3 \\ \underline{B}_3 \end{bmatrix} e^{i(\underline{K}\cdot\underline{r}-\omega t)} \quad \ldots(2.3)$$

+ Conjugate terms + higher order terms

where $\underline{K} = \underline{K}_1 - \underline{K}_2$ and $\omega = \omega_1 - \omega_2$

The equilibrium distribution function for the ions and electrons is represented by f_o^+ and f_o^- respectively. The first and second order perturbations to the distribution function are given by $f_{1,2}^{\pm}$ and $f_3^{\pm}$ respectively.

The zero'th order trajectory for the ions and electrons is given by

$$\frac{d\underline{v}'}{dt} = \varepsilon \underline{v}' \times \hat{z}\,\Omega$$

where $\underline{v}'$ = velocity vector at time t'

Ω = Ω_+ or Ω_-, depending upon the species being considered.

$\Omega_+ = \left|\frac{eB_o}{m_+c}\right|$

$\Omega_- = \left|\frac{eB_o}{m_-c}\right|$

ε = sign of charge on species being considered.

The solution to the above equation may be obtained by setting $u^{\pm} = v_x' \pm i\varepsilon v_y'$ and applying the boundary condition that v' = v for t' = t. This gives

$$v_x' = v_x \cos(\Omega t - \Omega t') - \varepsilon v_y \sin(\Omega t - \Omega t')$$

$$v_y' = \varepsilon v_x \sin(\Omega t - \Omega t') + v_y \cos(\Omega t - \Omega t') \qquad \ldots(2.4)$$

$$v_z' = v_z$$

The velocity components in equation (2.4) may be integrated, and with the use of the boundary condition that $\underline{r}' = \underline{r}$ for $t' = t$,

$$x' = -\frac{v_x}{\Omega} \sin \Omega\tau' + \frac{\varepsilon v_y}{\Omega}(1 - \cos\Omega\tau) + x$$

$$y' = -\frac{\varepsilon v_x}{\Omega}(1 - \cos\Omega\tau) - \frac{v_y}{\Omega}\sin\Omega\tau + y$$

$$z' = -v_z\tau + z \qquad \ldots(2.5)$$

$$\tau = t - t'$$

The collisionless Boltzmann equation given by equation (2.1), together with equations (2.2) and (2.3) may be used to express the change in the equilibrium distribution function along the zero'th order trajectory of a particle as follows:

$$\left(\frac{df_o^{\pm}}{dt}\right)_o = 0$$

The most general solution to the above equation is of the type

$$f_o^{\pm} = f_o^{\pm}\left(\frac{v_\perp^2}{2}, v_z\right) \qquad \ldots(2.6)$$

where $\perp$ and z refer to directions perpendicular and parallel to the static magnetic field, and $v_\perp^2 = v_x^2 + v_y^2$.

For the specific case considered in this thesis, $f_o^{\pm}$ will be taken to be Maxwellian, with equal temperatures in the directions perpendicular and parallel to the static magnetic field. That is,

$$f_o^{\pm} = \left(\frac{m_\pm}{2\pi K T^\pm}\right)^{3/2} \exp\left(\frac{-m_\pm}{2KT^\pm}\left(v_\perp^2 + v_z^2\right)\right) \qquad \ldots(2.7)$$

The velocity gradient of $f_o^{\pm}$ may be expressed as follows:

$$\frac{\partial f_o^{\pm}}{\partial v_x} = v_x f_{o\perp}^{\pm}$$

$$\frac{\partial f_o^{\pm}}{\partial v_y} = v_y f_{o\perp}^{\pm}$$

$$\frac{\partial f^{\pm}}{\partial v_z} = f_{oz}^{\pm} \qquad \ldots(2.8)$$

where $$f_{o\perp}^{\pm} = \frac{\partial f_o^{\pm}}{\partial\left(\frac{v_\perp^2}{2}\right)}$$

By using equation (2.4), it may be shown that $f_o^{\pm}$, $f_{o\perp}^{\pm}$, and $f_{oz}^{\pm}$ will have the same form in primed and unprimed coordinates. This is the result of $(v_\perp')^2$ and v_z' being constants of the zero order motion.

2.2 Solution for the Incident Fields

Through the use of equation (2.1) to (2.3), the equations governing the first order perturbations induced in the plasma by

the incident fields, may be expressed as

$$\left(\frac{df_1^{\pm}}{dt}\right)_0 = \left[-\frac{Z_{\pm}e\mathcal{E}_{\pm}}{m_{\pm}}\left(\underline{E}_1 + \frac{\underline{v}\times\underline{B}_1}{c}\right)\cdot\frac{\partial f_0^{\pm}}{\partial \underline{v}}\right]e^{i(\underline{k}_1\cdot\underline{r} - \omega_1 t)} \qquad ...(2.9)$$

plus a similar equation for $f_2^{\pm}$

Through the use of the following Maxwell's equation

$$\nabla \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t}$$

the integration of equation (2.9) along the zero order trajectory gives

$$f_1^{\pm}(\underline{r}, \underline{v}, t) = -\frac{Z_{\pm}e\mathcal{E}_{\pm}}{m_{\pm}}\int_{-\infty}^{t}\underline{E}_1\, e^{i(\underline{k}_1\cdot\underline{r}' - \omega_1 t')}$$

$$\left[1 + \frac{\cdot\underline{v}'\underline{k}_1}{\omega_1} - \frac{(\underline{v}'\cdot\underline{k}_1)}{\omega_1}\right]\cdot\frac{\partial f_0^{\pm}}{\partial \underline{v}'}\,dt' \qquad ...(2.10)$$

The first order current in the plasma may be obtained by taking the velocity moments of $\underline{v}\, f_1^{\pm}$. The dispersion relation for the first incident wave may then be obtained by substituting the resulting expression for the first order current into Maxwell's equations. A similar analysis is applicable for the second incident wave.

2.2.1 Dispersion Relation for the Incident Waves

The results given in Stix[10] (Chapter 9) may be used to express the first order current as,

$$\underline{J}_i = n_o e \left(\langle \underline{v} f_i^+ \rangle - \langle \underline{v} f_i^- \rangle \right) = \sum_K \frac{\varepsilon_K \omega_{PK}^2 \underline{\underline{M}}_i^K}{4\pi} \cdot E_i \qquad \ldots(2.11)$$

where $\langle \underline{v} f_i^{\pm} \rangle = \int_{-\infty}^{\infty} \underline{v} f_i^{\pm} \, d\underline{v}$

i = 1,2 and denotes the first and second incident waves respectively.

n_o = plasma density/cm^3

ω_{pk} = plasma frequency for species k

$\underline{\underline{M}}_i^k$ = mobility tensor for species k

ε_k = sign of charge on species k

In the following analysis, only two species will be considered, namely ions and electrons (k = +, -).

If finite Larmor radius effects are neglected, as well as collisionless damping and finite temperature effects

$$\underline{\underline{M}}_i^{\pm} = \begin{bmatrix} \frac{i\varepsilon_\pm \Omega_\pm}{2}\left(\frac{1}{\omega_i + \Omega_\pm} + \frac{1}{\omega_i - \Omega_\pm} \right) & \frac{\Omega_\pm}{2}\left(\frac{1}{\omega_i + \Omega_\pm} - \frac{1}{\omega_i - \Omega_\pm} \right) & 0 \\ -\frac{\Omega_\pm}{2}\left(\frac{1}{\omega_i + \Omega_\pm} - \frac{1}{\omega_i - \Omega_\pm} \right) & \frac{i\varepsilon_\pm \Omega_\pm}{2}\left(\frac{1}{\omega_i + \Omega_\pm} + \frac{1}{\omega_i - \Omega_\pm} \right) & 0 \\ 0 & 0 & \frac{i\varepsilon_\pm \Omega_\pm}{\omega_i} \end{bmatrix} \qquad \ldots(2.12)$$

Equations (2.11) and (2.12) may be used to express the total first order currents as follows:

$$J_{ix} = \frac{i a_i^1}{4\pi} E_{ix} + \frac{a_i^2}{4\pi} E_{iy}$$

$$J_{iy} = -\frac{a_i^2}{4\pi} E_{ix} + i \frac{a_i^1}{4\pi} E_{iy}$$

$$J_{iz} = \frac{i\omega_p^2}{4\pi\omega_i} E_{iz} \qquad \ldots(2.13)$$

where

$$a_i^1 = \frac{\omega_i \omega_p^2}{2}\left[\frac{1}{(\omega_i + \Omega_-)(\omega_i - \Omega_+)} + \frac{1}{(\omega_i - \Omega_-)(\omega_i + \Omega_+)}\right]$$

$$a_2^i = \frac{\omega_i \omega_p^2}{2}\left[\frac{-1}{(\omega_i + \Omega_-)(\omega_i - \Omega_+)} + \frac{1}{(\omega_i - \Omega_-)(\omega_i + \Omega_+)}\right]$$

$$\omega_p^2 = \omega_{p_-}^2 \left(1 + \frac{m_-}{m_+}\right)$$

By using equation (2.3) and Maxwell's equations

$$\nabla \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t}$$

$$\nabla \times \underline{B} = \frac{4\pi}{c}\underline{J} + \frac{1}{c}\frac{\partial \underline{E}}{\partial t} \qquad \ldots(2.14)$$

it may be shown that

$$\left[\left(\frac{\omega_i}{c}\right)^2 - K_i^2\right] \underline{E}_i = -i\frac{4\pi}{c}\left(\frac{\omega_i}{c}\right)\underline{J}_i - (\underline{K}_i \cdot \underline{E}_i)\underline{K}_i$$

Through the use of the continuity equation and Poisson's equation, the above equation may be expressed as follows:

$$\left[\left(\frac{\omega_i}{c}\right)^2 - K_i^2\right]\underline{E}_i = -i\frac{4\pi}{\omega_i}\left[\left(\frac{\omega_i}{c}\right)^2 \underline{J}_i - (\underline{K}_i \cdot \underline{J}_i)\,\underline{K}_i\right] \qquad \ldots(2.15)$$

where $i = 1,2$

In solving equation (2.15) for the dispersion relation, it will be convenient to use the refractive index for the plasma, which is defined by

$$\underline{n}_i = \frac{c}{\omega_i}\underline{K}_i \qquad \ldots(2.16)$$

If the incident waves are assumed to be Whistler modes with $\Omega_+ < \omega_1 < \Omega_-$, the z-component of equation (2.15) gives

$$E_{iz} = -\frac{n_{ix}}{n_{iz}}\left(\frac{\omega_i}{\omega_p}\right)^2 \frac{1}{\left(1 - \frac{1}{n_{iz}^2}\right)\left(1 - \varepsilon_{2i}\right)}\left[\frac{a_i^1}{\omega_i}E_{ix} - i\frac{a_i^2}{\omega_i}E_{iy}\right] \qquad \ldots(2.17)$$

where $\varepsilon_{2i} = \left(\frac{\omega_i}{\omega_p}\right)^2 (1 - n_i^2)/(1 - (n_{iz})^2)$

The x and y components of equation (2.15) may now be expressed in matrix form as follows:

$$\begin{bmatrix} 1 - n_i^2 - \frac{a_i^1}{\omega_i}(1 + 2\varepsilon_i) & i\frac{a_2^i}{\omega_i}(1 + 2\varepsilon_i) \\ -i\frac{a_2^i}{\omega_i} & 1 - n_i^2 - \frac{a_i^1}{\omega_i} \end{bmatrix}\begin{bmatrix} E_{ix} \\ E_{iy} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \ldots(2.18)$$

where $\quad \mathcal{E}_i = \frac{1}{2}\left\{ \frac{\varepsilon_{2i}\, n_{ix}^2\, n_{iz}^2 + n_{ix}^2(1-\varepsilon_{2i})}{(n_{iz}^2-1)(1-\varepsilon_{2i})} \right\}$

A non-trivial solution for E_x^i and E_y^i implies that det [ ] = 0. This gives

$$(1-n_i^2)^2 - 2(1-n_i^2)(1+\mathcal{E}_i)\frac{a_i^1}{\omega_i} + \left[\left(\frac{a_i^1}{\omega_i}\right)^2 - \left(\frac{a_i^2}{\omega_i}\right)^2\right](1+2\varepsilon_i) = 0 \qquad \ldots(2.19)$$

The solution of equation (2.19) will give the dispersion relation for the incident waves.

2.2.2 Reflections at the Plasma-Vacuum Interface

The assumed geometry of the plasma-vacuum interface is given in Figure (2.1). An oblique configuration is considered so that the results obtained may be applicable to the case where the effects of perturbations on the slope of the plasma-vacuum interface is studied. A similar calculation for the reflection coefficients has been made by Unz[19], Schmidt [20], and Sluijter[21].

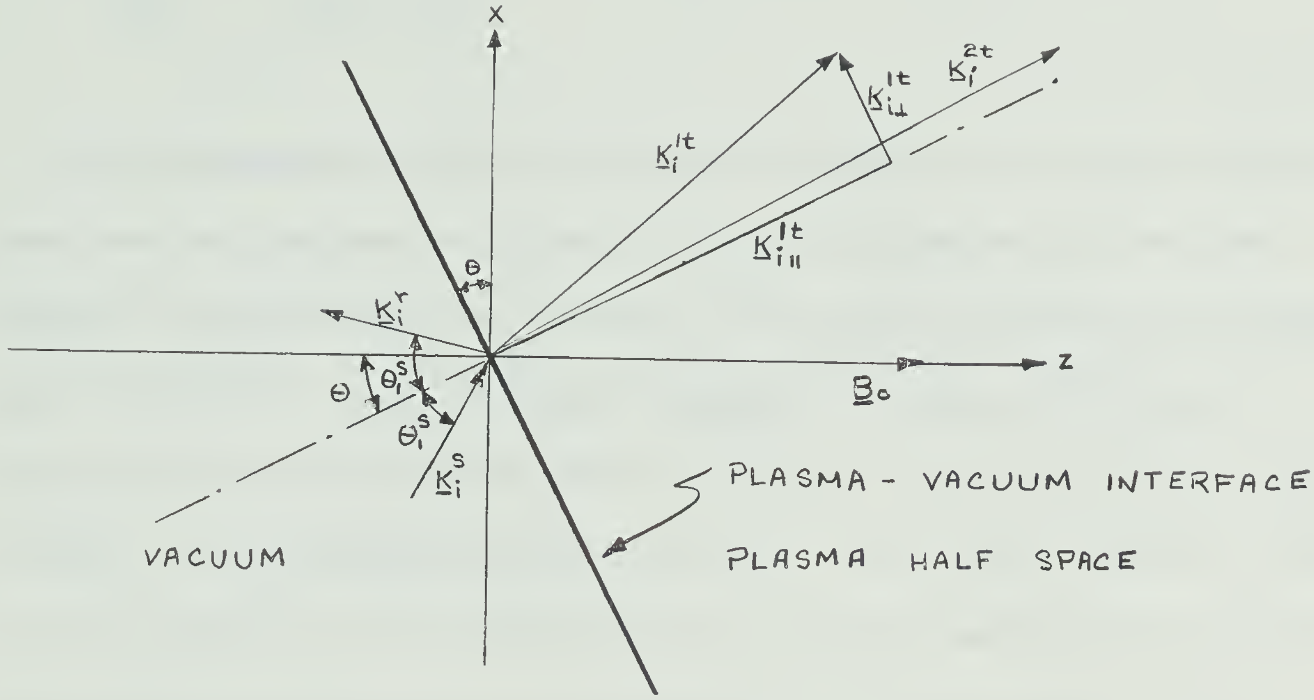

Figure 2.1 Geometry of Plasma-Vacuum Interface.

The notation used in Figure (2.1) is as follows:

$\underline{K}_i^r$ = propagator for the reflected wave

$\underline{K}_i^s$ = propagator for the incident wave

$\underline{K}_i^{1t}$ = propagator for the transmitted component with RH elliptical polarization

$\underline{K}_i^{2t}$ = propagator for the transmitted component with LH elliptical polarization

i = 1,2 and denotes the first and second incident waves respectively

The notation that will be used for the refractive indicies of the plasma to the waves given in Figure (2.1) will be that used for the propagation vectors, with η replacing $\underline{K}$. For example

$$\underline{n}_i^{1t} = c/\omega_i \; \underline{K}_i^{1t}$$

The transmitted portion of the wave incident on a plasma vacuum interface, as depicted in Figure (2.1) will decompose into two components, one right-hand and the other left-hand elliptically polarized (Rose & Clark[3]). If the incident frequency is taken in the range between the ion and electron gyro-frequencies, the LH polarized component will not be transmitted in the plasma. This is the result of the strong attenuation of the wave through cyclotron damping near the ion cyclotron frequency, and the negative value for the square of the refractive index for frequencies greater than the ion cyclotron frequency[22]. However, the RH elliptically polarized component, normally referred to as a Whistler mode may be made to penetrate the plasma by a suitable choice for the incident frequency and the static magnetic field. It is this component which will be considered in the nonlinear mixing process.

The following relations may be obtained from Figure (2.1):

$$\underline{K}_i^{s} = \left(\frac{\omega_i}{c}\right)\left[\sin(\theta + \theta_i^s),\ 0,\ \cos(\theta + \theta_i^s)\right]$$

$$\underline{K}_i^{r} = \left(\frac{\omega_i}{c}\right)\left[\sin(\theta_i^s - \theta),\ 0,\ -\cos(\theta_i^s - \theta)\right]$$

$$\underline{K}_i^{jt} = \left[K_{i\parallel}^{jt}\sin\theta + K_{i\perp}^{jt}\cos\theta,\ 0,\ K_{i\parallel}^{jt}\cos\theta - K_{i\perp}^{jt}\sin\theta\right] \qquad \ldots(2.20)$$

where $K_{i\perp}^{jt} = \left(\frac{\omega_i}{c}\right)\sin\theta_i^s$

$j = 1, 2$

The solution to equation (2.19) may then be written as follows:

$$(n_i^{jt})^2 = 1 - \left\{ \frac{a_i^1}{\omega_i}(1+\varepsilon_i^{jt}) + (3-2j)\frac{a_i^2}{\omega_i}\left[1 + 2\varepsilon_i^{jt} + \left(\frac{a_i^1}{a_i^2}\right)(\varepsilon_i^{jt})^2\right]\right\}$$

$$= (n_{i\parallel}^{jt})^2 + \sin^2\theta_1^s \qquad \ldots(2.21)$$

where

$$\varepsilon_i^{jt} = \frac{1}{2}\left\{\frac{\varepsilon_{2i}^{jt}\,(n_{ix}^{jt})^2\,(n_{iz}^{jt})^2 + (n_{ix}^{jt})^2\,(1-\varepsilon_{2i}^{jt})}{\left[(n_{iz}^{jt})^2-1\right]\left[1-\varepsilon_{2i}^{jt}\right]}\right\}$$

$$\varepsilon_{2i}^{jt} = \left(\frac{\omega_i}{\omega_p}\right)^2 \frac{\left[1-(n_i^{jt})^2\right]}{\left[1-(n_{iz}^{jt})^2\right]}$$

$j = 1,2$ and denotes waves with a RH and LH elliptically polarized electric field respectively.

For the cases that will be considered, $|\varepsilon_i^{jt}| << 1$. The solution for η_i^{jt} in equation (2.21) will be obtained through an iterative procedure. A first approximation to the solution for η_i^{jt} is obtained by setting ε_i^{jt} to zero. An approximate value for ε_i^{jt} may be obtained by substituting the above solution for η_i^{jt} into the expression given by equation (2.21) for ε_i^{jt}. A better approximation to the solution for η_i^{jt} may be obtained by substituting the above value for ε_i^{jt} into the equation for $(\eta_i^{jt})^2$. The accuracy of the expressions for the dispersion relations may be improved by further iterations. In the limit that $(\omega_i/\omega_p)^2 <<1$, and if terms of order $(\varepsilon_i^{jt})^2/2$ are neglected, the dispersion relation for the incident waves in the plasma may be written as

$$(n_{i\parallel}^{jt})^2 = \cos^2\theta_i^s - (1+\varepsilon_i^{jt})\left(\frac{a_i^1}{\omega_i} + (3-2j)\frac{a_i^2}{\omega_i}\right) \qquad \ldots(2.22)$$

where the value for ε_i^{jt} is obtained by substituting

$$(n_{i\parallel}^{jt})^2 = \cos^2\theta_i^s - \left(\frac{a_i^1}{\omega_i} + (3-2j)\frac{a_i^2}{\omega_i}\right)$$

, into the expression given by equation (2.21) for ε_i^{jt}.

The required expressions for the propagation vectors may be obtained by substituting the results from equation (2.21) (or from equation (2.22) in the limit that $(\varepsilon_i^{jt})^2/2 \ll 1$) into equation (2.20).

The transmitted field components will now be expressed in terms of E_{1x}^{1t} and E_{1x}^{2t} by the use of equation (2.17) and (2.18).

$$E_{iz}^{jt} = \Pi_{iz}^{jt} E_{ix}^{jt}$$

$$E_{iy}^{jt} = \Pi_{iy}^{jt} E_{ix}^{jt} \qquad \text{...(2.23)}$$

where i = 1,2 and denotes the first and second incident waves respectively.

j = 1,2 and denotes waves with RH and LH elliptically polarized electric fields respectively.

$$\Pi_{iz}^{1t} = -\frac{n_{ix}^{1t}}{n_{iz}^{1t}} a_{iz}^{+}$$

$$\Pi_{iy}^{1t} = i(1 - \delta_i^{+})$$

$$\Pi_{iz}^{1t} = -\frac{n_{ix}^{2t}}{n_{iz}^{2t}} a_{iz}^{-}$$

$$\Pi_{iy}^{2t} = -i(1 - \delta_i^{-})$$

$$\delta_i^+ = 1 - \frac{1}{\left\{ \left(\frac{a_i^1}{a_i^2}\right) \varepsilon_i^{1t} + \left[1 + 2\,\varepsilon_i^{1t} + \left(\frac{a_i^1}{a_i^2}\right)^2 (\varepsilon_i^{1t})^2 \right]^{1/2} \right\}}$$

$$= \varepsilon_i^{1t} \left(1 + \frac{a_i^1}{a_i^2} \right) \Big/ \left(1 + \varepsilon_i^{jt} (1 + a_i^1/a_i^2) \right) \qquad (\text{for } |\varepsilon_i^{1t}| \ll 1)$$

$$\delta_i^- = \frac{1}{\left\{ \left(\frac{a_i^1}{a_i^2}\right) \varepsilon_i^{2t} - \left[1 + 2\,\varepsilon_i^{2t} + \left(\frac{a_i^1}{a_i^2}\right)^2 (\varepsilon_i^{2t})^2 \right]^{1/2} \right\}} + 1$$

$$= \varepsilon_i^{2t} (1 - a_i^1/a_i^2) \Big/ \left(1 + \varepsilon_i^{2t} (1 - a_i^1/a_i^2) \right) \qquad (\text{for } |\varepsilon_i^{2t}| \ll 1)$$

$$\Delta_i^- = (\omega_i + \Omega_-)(\omega_i - \Omega_+)$$

$$\Delta_i^+ = (\omega_i - \Omega_-)(\omega_i + \Omega_+)$$

$$a_{iz}^+ = \frac{\omega_i^2}{\left[1 - \frac{1}{(n_{iz}^{1t})^2} \right] \left(1 - \varepsilon_{2i}^{2t} \right)} \left[\frac{1 - \delta_i^+/2}{\Delta_i^+} + \frac{\delta_i^+/2}{\Delta_i^-} \right]$$

$$a_{iz}^- = \frac{\omega_i^2}{\left[1 - \frac{1}{(n_{iz}^{2t})^2} \right] \left(1 - \varepsilon_{2i}^{2t} \right)} \left[\frac{1 - \delta_i^-/2}{\Delta_i^-} + \frac{\delta_i^-/2}{\Delta_i^+} \right]$$

Through the use of Maxwell's equations, the magnetic field components associated with the incident waves may also be expressed in terms of E_{ix}^{1t} and E_{ix}^{2t}.

$$\underline{B}_i^{jt} = \frac{c}{\omega_i} \left(\underline{K}_i^{jt} \times \underline{E}_i^{jt} \right) \qquad \ldots(2.24)$$

where i = 1,2 and denotes the first and second incident waves respectively.

j = 1,2 and denotes waves with RH and LH elliptically polarized electric fields respectively.

The following relations may be obtained by the use of equations (2.23) and (2.24):

$$B^{jt}_{ix} = \beta^{jt}_{ix} \; E^{jt}_{ix}$$

$$B^{jt}_{iy} = \beta^{jt}_{iy} \; E^{jt}_{ix}$$

$$B^{jt}_{iz} = \beta^{jt}_{iz} \; E^{jt}_{ix} \qquad \ldots(2.25)$$

where

$$\beta^{jt}_{ix} = -\frac{c}{\omega_i} \; K^{jt}_{iz} \; \Pi^{jt}_{iy}$$

$$\beta^{jt}_{iy} = \frac{c}{\omega_i} \left(K^{jt}_{iz} - K^{jt}_{ix} \; \Pi^{jt}_{iz} \right)$$

$$\beta^{jt}_{iz} = \frac{c}{\omega_i} \; K^{jt}_{ix} \; \Pi^{jt}_{iy}$$

The incident waves may be taken as being transverse in free space, that is

$$(\underline{E}^{s}_{i} \cdot \underline{k}^{s}_{i}) = 0 \qquad \ldots(2.26)$$

where $\underline{K}^{s}_{i} = \omega_i/c \; (\sin\,(\theta + \theta^{s}_{i}),\; 0,\; \cos\,(\theta + \theta^{s}_{i}))$

= propagation vector for the i'th incident wave in free space.

i = 1,2

By using equations (2.24) and (2.26), the following relations may be derived for the magnetic field components in free space:

$$B^s_{ix} = -\cos(\theta + \theta^s_i)\, E^s_{iy}$$

$$B^s_{iy} = 1/\cos(\theta + \theta^s_i)\ E^s_{ix}$$

$$B^s_{iz} = \sin(\theta + \theta^s_i)\, E^s_{iy} \qquad \ldots(2.27)$$

Both E^s_{ix} and E^s_{iy} are known, since these are the chosen components for the incident waves. The electric field components perpendicular to the y-axis for the incident waves in free space will be denoted by $E^s_{i\perp}$. That is,

$$\underline{E}^s_{i\perp} = E^s_{ix}\, \hat{\varepsilon}_x + E^s_{iz}\, \hat{\varepsilon}_z \qquad \ldots(2.28)$$

By taking the reflected wave as being transverse and using Maxwell's equations to express the relationship between the magnetic and electric field components, the following relations may be obtained for the reflected wave.

$$\underline{K}^r_i = \frac{\omega_i}{c} \left\{ \sin(\theta^s_i - \theta),\ 0,\ -\cos(\theta^s_i - \theta) \right\}$$

$$
\begin{aligned}
E^{r}_{iz} &= \tan(\theta^{s}_{i} - \theta)\, E^{r}_{ix} \\
B^{r}_{ix} &= \cos(\theta^{s}_{i} - \theta)\, E^{r}_{iy} \\
B^{r}_{iy} &= -1/\cos(\theta^{s}_{i} - \theta)\, E^{r}_{ix} \\
B^{r}_{iz} &= \sin(\theta^{s}_{i} - \theta)\, E^{r}_{iy}
\end{aligned}
\qquad \ldots(2.29)
$$

A definition similar to that made in equation (2.28) for the incident waves is now made for the waves reflected from the plasma-vacuum interface. That is $\underline{E}^{r}_{i\perp}$ is defined as follows:

$$
\underline{E}^{r}_{i\perp} = E^{r}_{ix}\, \hat{\varepsilon}_{x} + E^{r}_{iz}\, \hat{\varepsilon}_{z} \qquad \ldots(2.30)
$$

The application of the tangential boundary conditions, namely, the continuity of the electric and magnetic field components tangential to the plasma-vacuum interface yields

$$
\begin{bmatrix}
\Pi^{1t}_{iy} & \Pi^{2t}_{iy} & 0 & -1 \\
\cos\theta - \Pi^{1t}_{iz}\sin\theta & \cos\theta - \Pi^{2t}_{iz}\sin\theta & -\cos\theta + \tan(\theta^{s}_{i} - \theta)\sin\theta & 0 \\
\beta^{1t}_{iy} & \beta^{2t}_{iy} & 1/\cos(\theta^{s}_{i} - \theta) & 0 \\
\beta^{1t}_{ix}\cos\theta - \beta^{1t}_{iz}\sin\theta & \beta^{2t}_{ix}\cos\theta - \beta^{2t}_{iz}\sin\theta & 0 & \begin{matrix} -\cos(\theta^{s}_{i}-\theta)\cos\theta \\ +\sin(\theta^{s}_{i}-\theta)\sin\theta \end{matrix}
\end{bmatrix}
$$

(continued on page 29)

$$\begin{bmatrix} E_{ix}^{1t} \\ E_{ix}^{2t} \\ E_{ix}^{r} \\ E_{iy}^{r} \end{bmatrix} = \begin{bmatrix} E_{iy}^{s} \\ E_{i\perp}^{s} \cos\theta_i^s \\ E_{i\perp}^{s} \\ -E_{iy}^{s} \cos\theta_i^s \end{bmatrix} \qquad \ldots(2.31)$$

The transmitted field components may be uncoupled from the reflected field components by combining the first and last equations and the second and third equations obtained by the expansion of equation (2.31). By defining

$$\begin{aligned}
a_i &= \cos(\theta_i^s - \theta)\cos\theta - \sin(\theta_i^s - \theta) \\
a_i' &= \cos\theta - \tan(\theta_i^s - \theta)\sin\theta \\
N_{i1} &= \left(\cos\theta - \Pi_{iz}^{1t}\sin\theta\right)/a_i' + \cos(\theta_i^s - \theta)\beta_{iy}^{1t} \\
N_{i2} &= \left(\cos\theta - \Pi_{iz}^{2t}\sin\theta\right)/a_i' + \cos(\theta_i^s - \theta)\beta_{iy}^{2t} \\
N_{i3} &= \Pi_{iy}^{1t} - \left(\beta_{ix}^{1t}\cos\theta - \beta_{iz}^{1t}\sin\theta\right)/a_i \\
N_{i4} &= \Pi_{iy}^{2t} - \left(\beta_{ix}^{2t}\cos\theta - \beta_{iz}^{2t}\sin\theta\right)/a_i \\
N_{i5} &= \cos(\theta_i^s - \theta)\left(1 + 1/a_i\right) \\
N_{i6} &= 1 + \cos(\theta_i^s - \theta)/a_i
\end{aligned} \qquad \ldots(2.32)$$

the equations governing the transmitted field components may be expressed as follows:

$$\begin{bmatrix} N_{i1} & N_{i2} \\ N_{i3} & N_{i4} \end{bmatrix} \begin{bmatrix} E_{ix}^{1t} \\ E_{ix}^{2t} \end{bmatrix} = \begin{bmatrix} N_{i5} \\ N_{i6} \end{bmatrix} \begin{bmatrix} E_{i\perp}^{s} \\ E_{iy}^{s} \end{bmatrix} \qquad \ldots(2.33)$$

The solution to the above equation may be readily obtained through the use of Cramer's Rule.

$$E_{ix}^{1t} = \frac{\left(N_{i5} N_{i4} E_{i\perp}^{s} - N_{i6} N_{i2} E_{iy}^{s} \right)}{DET_N}$$

$$E_{ix}^{2t} \quad \frac{\left(N_{i6} N_{i1} E_{iy} - N_{i5} N_{i3} E_{i\perp}^{s} \right)}{DET_N} \qquad \ldots(2.34)$$

where $DET_N = N_{i1} N_{i4} - N_{i2} N_{i3}$

The reflected field components may be found by substituting the results obtained in equation (2.34) into equation (2.31). This gives

$$E_{ix}^{r} = E_{i\perp}^{s} \cos(\theta_i^s - \theta) - \beta_{iy}^{1t} \cos(\theta_i^s - \theta) E_{ix}^{1t} - \beta_{iy}^{2t} \cos(\theta_i^s - \theta) E_{ix}^{2t}$$

$$E_{iy}^{r} = - E_{iy}^{s} + \Pi_{iy}^{1t} E_{ix}^{1t} + \Pi_{iy}^{2t} E_{ix}^{2t} \qquad \ldots(2.35)$$

With the help of equation (2.3), the percentage of the power in an incident wave that is reflected at the plasma-vacuum interface may be shown to be equal to $100 \left\{ \dfrac{|E_{i\perp}^{r}|^2 + |E_{iy}^{r}|^2}{|E_{i\perp}^{s}|^2 + |E_{iy}^{s}|^2} \right\}$.

The normal configuration that will be assumed for the incident waves and the plasma-vacuum interface is depicted in Figure (2.2). The above relation for the reflected power will be used when examples are considered in Chapter 3.

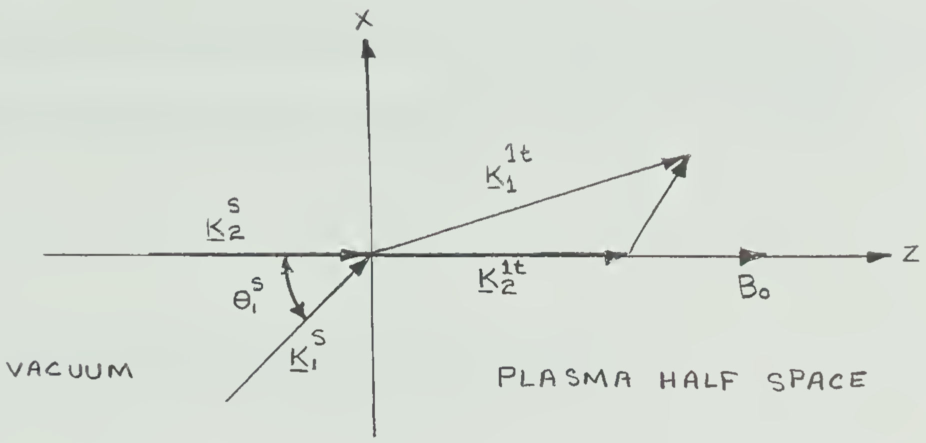

FIGURE 2.2. Wave Vectors and Plasma-Vacuum Interface. Only the waves considered in the mixing process are shown in this Figure. The two strongly attenuated LH polarized modes have not been shown.

A resonance in the magnitude of the fields associated with the mixed wave may be realized if the propagation vector of the mixed wave approaches that of a natural mode in the plasma. In this particular case, if the mixed wave is taken as propagating approximately perpendicular to $\underline{B}_o$, the natural mode is an extra-ordinary Alfven wave. If a field resonance is to occur in the plasma for the case that $\underline{K}_2 \parallel \underline{B}_o$, the required angle of incidence for the first incident wave may be approximated as follows by the use of the results given in Spitzer[23] for the dispersion relation for an extra-ordinary Alfven wave:

$$\theta_1^s \sim \sin^{-1} \frac{\omega}{\omega_1}\left[1 - \frac{\omega_p^2}{\omega^2 - \Omega_+\Omega_- + \dfrac{\omega^2\Omega_-^2\left(1 - \frac{m_-}{m_+}\right)^2}{\omega_p^2 - \omega^2 + \Omega_+\Omega_-}}\right] \qquad \ldots(2.36)$$

where $\omega_p^2 = \omega_{p-}^2\,(1 + m_-/m_+)$

The propagation vector for the difference frequency wave (See Figure

(2.2)) is given by

$$\underline{K} = \underline{K}_1^{1t} - \underline{K}_2^{1t} \qquad \ldots(2.37)$$

The object of this thesis is to investigate the use of this difference frequency wave for the heating of ions in a plasma.

2.3 Second Order Fields And Currents

In this section, the second order fields and currents will be obtained. Through the use of equation (2.1), (2.2), and (2.3), the equation governing the second order perturbations may be expressed as

$$\begin{aligned}\left(\frac{df_3^{\pm}}{dt}\right)_0 = & -\frac{Z_{\pm} e \mathcal{E}_{\pm}}{m_{\pm}}\left(\underline{E}_3 + \frac{\underline{v}\times\underline{B}_0}{c}\right)\cdot\frac{\partial f_0^{\pm}}{\partial \underline{v}} \\ & -\frac{Z_{\pm} e \mathcal{E}_{\pm}}{m_{\pm}}\left(\underline{E}_2^{*} + \frac{\underline{v}\times\underline{B}_2^{*}}{c}\right)\cdot\frac{\partial f_1^{\pm}}{\partial \underline{v}} \\ & -\frac{Z_{\pm} e \mathcal{E}_{\pm}}{m_{\pm}}\left(\underline{E}_1 + \frac{\underline{v}\times\underline{B}_1}{c}\right)\cdot\frac{\partial f_2^{\pm *}}{\partial \underline{v}}\end{aligned} \qquad \ldots(2.38)$$

The velocity moments of $f_3^{\pm}$ will give rise to three terms. The first term will be an induced component from which the ion energy absorption will be calculated. The remaining two terms correspond to driving terms. The contributions to the second order distribution function will be denoted as follows:

$$f_3^{\pm} = f_{33}^{\pm} + f_{32}^{\pm} + f_{31}^{\pm} \qquad \ldots(2.39)$$

The first, second and third terms on the RHS of equation (2.39) will denote the resulting contributions to $f_3^{\pm}$ from the first, second, and third terms respectively, on the RHS of equation (2.38).

The procedure that will be used to solve for the various com-

ponents of $f_3^{\pm}$ given in equation (3.39) is outlined below for $f_{32}^{\pm}$. Specific reference to a particular species will not be made since necessary signs and factors are preserved in terms as ε, Z, and m. From equation (2.38)

$$f_{32} = \lim_{T \to \infty} -\frac{Ze\varepsilon}{m} \int_{-T}^{t} \left(E_2^* + \frac{v \times B_2^*}{c} \right) \cdot \frac{\partial f_1}{\partial v} \, dt' \qquad \ldots(2.40)$$

By using the expression given for f_1 in equation (2.10), the above equation may be expressed in the following form:

$$f_{32} = \int_{\tau} \int_{\tau'} d\tau \, d\tau' \, G\left(E_2^* (\underline{v}, \tau) \, E_1(\underline{r}, \tau'), \underline{v} \right) \qquad \ldots(2.41)$$

where

with the integration limits on τ and τ' being $[0,\infty)$ and $[0,\infty]$ respectively. The derivation of equation (2.41) is given in Appendix B.

2.3.1 Velocity Moments of f_{32}

Following the procedure outlined in equation (2.38) to (2.41) and Appendix B, the first step in solving for f_{32} is to find an expression for $f_1(t')$. This may be obtained from equation (2.10). By using equation (2.5), the resulting phase term, $i(\underline{K}_1 \cdot \underline{r}'' - \omega_1 t'')$ may be expressed in terms of the variables at $\underline{r}'$ and t'. That is

$$i\underline{K}_1 \cdot r'' - i\omega_1 t'' = i\underline{K}_1 \cdot r' - i\omega_1 t' + i\frac{v_x'}{\Omega}\left(-K_{1x}\sin\Omega\tau'\right)$$

$$+ \frac{i\varepsilon v_y'}{\Omega} K_{1x}\left(1-\cos\Omega\tau'\right) + i v_z'\left(-K_{1z}\tau'\right) + i\omega_1\tau' \qquad \ldots(2.42)$$

where $\tau' = t' - t''$

By performing a similar analysis for the term

$$\underline{E}_1 e^{i(K_1\cdot r'' - \omega_1 t)}\left[1 + \cdot\frac{\underline{v}''\underline{K}_1}{\omega_1} - (\underline{v}'\cdot\underline{K}_1)\right]\cdot\frac{\partial f_o(\underline{v}'')}{\partial\underline{v}''}$$

$f_1(\underline{r}',\underline{v}',t')$ may be expressed as follows:

$$f_1(r',v',t') = -\frac{Ze\varepsilon}{m} e^{i(\underline{K}_1\cdot\underline{r}' - \omega_1 t')}\int_0^\infty d\tau\, e^{i\left[-\frac{v_x'}{\Omega}K_{1x}\sin\Omega\tau' + v_y'\varepsilon\frac{K_{1x}}{\Omega}(1-\cos\Omega\tau')\right.}$$

$$\left.+ v_z'(-K_{1z}\tau') + \omega\tau'\right]\left\{E_{1x}\left(v_x'\cos\Omega\tau' - \varepsilon v_y'\sin\Omega\tau'\right)\left[f_{o\perp} + \frac{K_{1z}}{\omega_1}\left(f_{oz} - v_z f_{o\perp}\right)\right]\right.$$

$$+ E_{1y}\left(\varepsilon v_x'\sin\Omega\tau' + v_y'\cos\Omega\tau'\right)\left[f_{o\perp} + \frac{K_{1z}}{\omega_1}\left(f_{oz} - v_z' f_{o\perp}\right)\right]$$

$$\left.+ E_{1z}\left[\frac{v_x'}{\omega_1}\left(-K_{1x}\cos\Omega\tau'\right) + \frac{v_y'}{\omega_1}\varepsilon K_{1x}\sin\Omega\tau'\right]\left(f_{oz} - v_z' f_{o\perp}\right) + E_{1z} f_{oz}\right\}$$

From the results given by equation (2.8) for a plasma with a Maxwellian equilibrium distribution, and with equal temperatures in the directions perpendicular and parallel to the static magnetic field, it is easy to show that $(f_{oz} - v_z' f_{o\perp}) = 0$. For this particular case, the above expression for $f_1(\underline{r}',\underline{v}',t')$ simplifies to

$$f_1(r', v', t') = -\frac{Ze\varepsilon}{m} e^{i(\underline{K_1}\cdot \underline{r}' - \omega_1 t')} \int_0^\infty d\tau' \, e^{i\left[-v_x' \frac{K_{1x}}{\Omega} \sin \Omega\tau' + v_y' \varepsilon \frac{K_{1x}}{\Omega}(1-\cos\Omega\tau') - v_z' K_{1z}\tau' + \omega_1 \tau'\right]} \left\{ E_{1x}\left(v_x' \cos\Omega\tau' - \varepsilon v_y \sin\Omega\tau'\right) f_{0\perp} + E_{1y}\left(\varepsilon v_x' \sin\Omega\tau' + v_y' \cos\Omega\tau'\right) f_{0\perp} + E_{1z} f_{0z}\right\}$$

...(2.43)

By using that $\underline{B}_2^* = c/w_2 \underline{K}_2 \times \underline{E}_2^*$, equation (2.40) may be expressed as follows:

$$f_{32} = -\frac{Ze\varepsilon}{m}\int_{-\infty}^{t} \underline{E}_2^* \, e^{i(\underline{K}_2\cdot\underline{r}' - \omega_2 t')} \left[1 + \cdot\underline{v}'\frac{\underline{K}_2}{\omega_2} - \left(\frac{\underline{v}'\cdot\underline{K}_2}{\omega_2}\right)\right] \cdot \frac{\partial f_1(r', v', t')}{\partial \underline{v}'}$$

...(2.44)

where $f_1(\underline{r}', \underline{v}', t')$ is given by equation (2.43).

If only the transmitted incident waves with a RH elliptically polarized electric field are considered in the mixing process, the following relations may be obtained from equation (2.23):

$$E_{1y} = i(1 - \delta_1^+) E_{1x}$$
$$E_{2y}^* = -i E_{2x}^*$$

...(2.45)

In deriving equation (2.45), the propagation vector for the second incident wave was assumed to be parallel to the static magnetic field in the plasma (see Figure 2.2). Through the use of equations (2.43) and (2.45), the partial derivative of $f_1(\underline{r}', \underline{v}', t')$ with respect to $\underline{v}'$ may be shown to be equal to

$$\frac{\partial f_1(\underline{r}',\underline{v}',t')}{\partial \underline{v}'} = -\frac{Ze\varepsilon}{m}\, e^{i(\underline{K}_1\cdot\underline{r}'-\omega_1 t')}\, f_0\, E_{1x}\int_0^\infty d\tau'\,\exp i\Big[-\frac{v_x}{\Omega}K_{1x}\sin\Omega\tau'$$
$$+ v_y'\,\varepsilon\frac{K_{1x}}{\Omega}(1-\cos\Omega\tau') - v_z' K_{1z}\tau' + \omega_1\tau\Big]\Big\{\Big[v_+'\, e^{i\varepsilon\Omega\tau'} - i\delta_1^+\big(\varepsilon v_x'\sin\Omega\tau$$
$$+ v_y'\cos\Omega\tau'\big) + v_z'\Big(\frac{E_{1z}}{E_{1x}}\Big)\Big]\Big[\hat{\varepsilon}_x\Big(\frac{m}{KT}v_x' + i\frac{K_{1x}}{\Omega}\sin\Omega\tau'\Big) + \hat{\varepsilon}_y\Big(\frac{m}{KT}v_y'$$
$$-i\varepsilon\frac{K_{1x}}{\Omega}(1-\cos\Omega\tau')\Big) + \hat{\varepsilon}_z\Big(\frac{m}{KT}v_z' + iK_{1z}\tau'\Big)\Big] - \hat{\varepsilon}_x\big(e^{i\varepsilon\Omega\tau'} - i\delta_1^+\varepsilon\sin\Omega\tau'\big)$$
$$-i\varepsilon v_y\big(e^{i\varepsilon\Omega\tau'} - \delta_1^+\cos\Omega\tau'\big) - \hat{\varepsilon}_z\Big(\frac{E_{1z}}{E_{1x}}\Big)\Big\} \qquad \ldots(2.46)$$

where $v_+' = v_x' + iv_y'$

By the use of equations (2.4) and (2.5), the variables in equations (2.44) and (2.46) may be expressed in terms of $\underline{r}$, $\underline{v}$, and τ, where $\tau = t - t'$. Equations (2.44) and (2.46) may then be combined to give the following equation for f_{32}:

$$f_{32} = \frac{Z^2e^2}{m^2}E_{1x}E_{2x}^*\int_0^\infty\!\!\int_0^\infty d\tau\, d\tau'\, f_0\,\exp i(\underline{K}\cdot\underline{r}-\omega t)\,\exp i\Big[-\frac{K_x}{\Omega}v_x\sin\Omega(\tau+\tau')$$
$$+\frac{K_x}{\Omega}\varepsilon v_y\big(1-\cos\Omega(\tau+\tau')\big) + \tau(\omega - K_z v_z) + \tau'(\omega_1 - K_{1z})\Big]\Big\{\Big[v_+\, e^{i\varepsilon\Omega(\tau+\tau')}$$
$$-\frac{\varepsilon\delta_1^+}{2}\big(v_\varepsilon e^{i\Omega(\tau+\tau')} - v_{-\varepsilon}e^{-i\Omega(\tau+\tau')}\big) + v_z\Big(\frac{E_{1z}}{E_{1x}}\Big)\Big]\Big[\Big(1-\frac{K_2 v_z}{\omega_2}\Big)\Big(\frac{m}{KT}v_- e^{-i\varepsilon\Omega\tau}$$
$$+\frac{\varepsilon K_x}{\Omega}\big(e^{i\varepsilon\Omega\tau'}-1\big)\Big) + v_- e^{-i\varepsilon\Omega\tau}\frac{K_2}{\omega_2}\Big(\frac{m}{KT}v_z + iK_{1z}\tau'\Big)\Big] - \Big(1-\frac{K_2}{\omega_2}v_z\Big)$$
$$\big(2e^{i\varepsilon\Omega\tau'} - \delta_1^+ e^{i\varepsilon\Omega\tau'}\big) - v_- e^{-i\varepsilon\Omega\tau}\frac{K_2}{\omega_2}\Big(\frac{E_{1z}}{E_{1x}}\Big)\Big\} \qquad \ldots(2.47)$$

where $K_x = K_{1x}$ (for $\underline{K}_2 \parallel \underline{B}_o$, see Figure (2.2))

$v_+ = v_x + iv_y$

$v_- = v_x - iv_y$

$\omega = \omega_1 - \omega_2$ = difference frequency

The next step in the analysis will be to express the integrand in equation (2.47) in terms of the various powers of the velocity components v_x and v_y. Equation (2.47) may then be expressed as:

$$f_{32} = \frac{Z^2 e^2}{m^2} E_{1x} E_{2x} e^{i(K\cdot r - \omega t)} \left(\frac{m}{2\pi KT}\right) \int_0^\infty \int_0^\infty d\tau\, d\tau'\, g \,(\alpha_{21} v_x^2 + \alpha_{22} v_y^2$$

$$+ i\alpha_{23} v_x v_y + \alpha_{24} v_x + i\alpha_{25} v_y + \alpha_{26}) \exp\left[i a v_x - \frac{m v_x^2}{2KT} + i b v_y \right.$$

$$\left. - \frac{m v_y^2}{2KT} + i\psi \right] \qquad \ldots (2.48)$$

where

$$\delta_1 = \left(1 + i \frac{KT}{m} \frac{K_2 K_{1z}}{\omega_2} \tau'\right)$$

$$\gamma_1 = \left(1 - \frac{K_2 v_z}{\omega_2}\right)\left(e^{i\varepsilon\Omega\tau'} - 1\right)$$

$$\alpha_{21} = \frac{m}{KT} \delta_1 \left[e^{i\varepsilon\Omega\tau'} - \frac{\varepsilon \delta_1^+}{2} \left(e^{i\Omega[(1-\varepsilon)\tau + \tau']} - e^{-i\Omega[(1+\varepsilon)\tau + \tau']} \right) \right]$$

$$\alpha_{22} = \frac{m}{KT} \delta_1 \left[e^{i\varepsilon\Omega\tau'} - \frac{\delta_1^+}{2} \left(e^{i\Omega[(1-\varepsilon)\tau + \tau']} + e^{-i\Omega[(1+\varepsilon)\tau + \tau']} \right) \right]$$

$$\alpha_{23} = -\frac{m}{KT} \frac{\delta_1^+}{2} \delta_1 \left[(1-\varepsilon) e^{i\Omega[(1-\varepsilon)\tau + \tau']} + (1+\varepsilon) e^{-i\Omega[(1+\varepsilon)\tau + \tau']} \right]$$

$$\alpha_{24} = \frac{\varepsilon \gamma_1 K_x}{\Omega} \left[e^{i\varepsilon\Omega(\tau+\tau')} - \frac{\varepsilon \delta_1^+}{2} \left(e^{i\Omega(\tau+\tau')} - e^{-i\Omega(\tau+\tau')} \right) \right]$$

$$+ \left(\delta_1 \frac{m}{KT} v_z - \frac{K_2}{\omega_2} \right) \left(\frac{E_{1z}}{E_{1x}} \right) e^{-i\varepsilon\Omega\tau}$$

$$\alpha_{25} = \frac{\varepsilon \gamma_1 K_x}{\Omega} \left[e^{i\varepsilon\Omega(\tau+\tau')} - \frac{\delta_1^+}{2} \left(e^{i\Omega(\tau+\tau')} + e^{-i\Omega(\tau+\tau')} \right) \right]$$

$$- \left(\delta_1 \frac{m}{KT} v_z - \frac{K_2}{\omega_2} \right) \left(\frac{E_{1z}}{E_{1x}} \right) e^{-i\varepsilon\Omega\tau}$$

$$\alpha_{26} = \left(1 - \frac{K_2 v_z}{\omega_2}\right) \left[-(2 - \delta_1^+) e^{i\varepsilon\Omega\tau'} + \frac{\varepsilon K_x}{\Omega} \left(\frac{E_{1z}}{E_{1x}}\right) v_z \left(e^{i\varepsilon\Omega\tau'} - 1\right) \right]$$

$$g = \left(\frac{m}{2\pi KT}\right)^{1/2} \exp\left(-\frac{m v_z^2}{2KT}\right)$$

$$a = -\frac{K_x}{\Omega}\sin\Omega(\tau+\tau')$$

$$b = \frac{\varepsilon K_x}{\Omega}\left(1-\cos\Omega(\tau+\tau')\right)$$

$$\psi = (\omega - K_z v_z)\tau + (\omega_1 - K_{1z} v_z)\tau'$$

In the next step, an integration over the velocity components v_x and v_y will be performed for the various velocity moments of f_{32}. Following the notation used in Stix[10], the velocity moment for $q(\underline{v})$ in the x-y velocity space is:

$$\langle q(\underline{v})\rangle_\perp = \iint_{-\infty}^{\infty} f_{32}\, q(\underline{v})\, dv_x\, dv_y \qquad \ldots(2.50)$$

Note that the expression for f_{32} given by equation (2.48) still requires an integration over τ and τ'. This will be performed later in the analysis.

In the integration of equation (2.48) over v_x and v_y, integrals of the following type appear (see Stix[10], pp 173):

$$G(p,a) = \left(\frac{m}{2\pi KT}\right)^{1/2}\int_{-\infty}^{\infty} v^p \exp\left(iav - \frac{mv^2}{2KT}\right) dv \qquad \ldots(2.51)$$

By completing the square in the exponential,

$$G(p,a) = \left(\frac{m}{2\pi KT}\right)^{1/2}\int_{-\infty}^{\infty} v^p e^{-a^2\frac{KT}{m}} e^{-\frac{m}{2KT}\left(v - ia\frac{KT}{m}\right)^2} dv$$

A standard integral form is obtained by defining $u = v - i\,\frac{akT}{m}$. The

results for p = 0, 1, 2, and 3 are as follows:

$$G(0,a) = \exp -\left(\frac{a^2 KT}{2m}\right)$$

$$G(1,a) = \frac{i a KT}{m} \exp -\left(\frac{a^2 KT}{2m}\right)$$

$$G(2,a) = \left(\frac{KT}{m} - \frac{a^2 K^2 T^2}{m^2}\right) \exp -\left(\frac{a^2 KT}{2m}\right) \qquad \ldots(2.52)$$

$$G(3,a) = ia\left(\frac{KT}{m}\right)^2 \left(3 - a^2 \frac{KT}{m}\right) \exp -\left(\frac{a^2 KT}{2m}\right)$$

When an integration over v_x and v_y is performed, it is evident from equation (2.52) that an exponential term with the exponent $-(a^2 + b^2)KT/2m$ will appear for all values of p. This exponent may be expanded by the use of the definitions for a and b to give

$$-(a^2 + b^2)\frac{KT}{m} = -\lambda\left(1 - \cos\Omega(\tau + \tau')\right) \qquad \ldots(2.53)$$

where $\lambda = \dfrac{K_x^2 \, KT}{\Omega^2 \, m}$

The factor λ is proportional to the square of the ratio of the Larmor radius to the wavelength perpendicular to the static magnetic field. This factor represents the finite Larmor radius correction terms and will later appear as an expansion parameter. In the case being considered, namely an electromagnetic wave with a frequency somewhere in the range between the ion and electron cyclotron frequencies, incident on a plasma half-space, (See Figure (2.2)) the following inequality may be shown for λ^+:

$$\lambda^+ = \frac{K_x^2 \, KT^+}{\Omega_+^2 \, m_+} < 1.68 \times 10^{-10} \, C_1^2 \, T^+_{MAX} \sin^2 \theta_1^S \qquad \ldots(2.54)$$

where T^+ = ion temperature in degrees Kelvin

$\omega_1 = C_1 \Omega_H$

$\Omega_H = (\Omega_+ \Omega_-)^{1/2}$ = Hybrid frequency

If the ions and electrons are taken to be at the same temperature,

$$\lambda^- = \left(\frac{m_-}{m_+}\right) \lambda^+$$

By using the notation given in equation (2.62), the various velocity moments of f_{32} in the x-y velocity space may be expressed as follows:

$$\langle f_{32} \rangle_\perp = \frac{Z^2 e^2}{m KT} E_{1x} E_{2x}^* e^{i(\underline{K}\cdot\underline{r} - \omega t)} \int_0^\infty \int_0^\infty d\tau\, d\tau'\, g(v_z)\, e^{\phi} \left\{ \alpha_{21} \frac{KT}{m}\left(1 - a^2 \frac{KT}{m}\right) \right.$$
$$\left. + \alpha_{22} \frac{KT}{m}\left(1 - b^2 \frac{KT}{m}\right) - i\alpha_{23}\, ab \left(\frac{KT}{m}\right)^2 + i\,\alpha_{24}\, a \frac{KT}{m} - \alpha_{25}\, b \frac{KT}{m} + \alpha_{26} \right\}$$

$$\langle v_x f_{32} \rangle_\perp = \frac{Z^2 e^2}{m KT} E_{1x} E_{2x}^* e^{i(\underline{K}\cdot\underline{r} - \omega t)} \int_0^\infty \int_0^\infty d\tau\, d\tau'\, g(v_z)\, e^{\phi} \left\{ i\alpha_{21}\, a \left(\frac{KT}{m}\right)^2 \right.$$
$$\left(3 - \frac{a^2 KT}{m}\right) + i\,\alpha_{22}\, a \left(\frac{KT}{m}\right)^2 \left(1 - b^2 \frac{KT}{m}\right) - \alpha_{23}\, b \left(\frac{KT}{m}\right)^2 \left(1 - a^2 \frac{KT}{m}\right)$$
$$\left. + \alpha_{24} \frac{KT}{m}\left(1 - a^2 \frac{KT}{m}\right) - i\,\alpha_{25}\, ab \left(\frac{KT}{m}\right)^2 + i\,\alpha_{26}\, a \frac{KT}{m} \right\}$$

$$\langle v_y f_{32} \rangle_\perp = \frac{Z^2 e^2}{m KT} E_{1x} E_{2x}^* e^{i(\underline{K}\cdot\underline{r} - \omega t)} \int_0^\infty \int_0^\infty d\tau\, d\tau'\, g(v_z)\, e^{\phi} \left\{ i\alpha_{21}\, b \left(\frac{KT}{m}\right)^2 \right.$$
$$\left(1 - a^2 \frac{KT}{m}\right) + i\,\alpha_{22}\, b \left(\frac{KT}{m}\right)^2 \left(3 - b^2 \frac{KT}{m}\right) - \alpha_{23}\, a \left(\frac{KT}{m}\right)^2 \left(1 - b^2 \frac{KT}{m}\right)$$
$$\left. - \alpha_{24}\, ab \left(\frac{KT}{m}\right)^2 + i\,\alpha_{25} \left(\frac{KT}{m}\right)\left(1 - b^2 \frac{KT}{m}\right) + i\alpha_{26}\, b \left(\frac{KT}{m}\right) \right\}$$

...(2.55)

where $\phi = -\lambda\,(1 - \cos\Omega(\tau + \tau')) + i(\omega - K_z v_z)\tau + i(\omega_1 - K_{1z} v_z)\tau'$

The term $e^{\lambda\cos\Omega(\tau + \tau')}$ will be expanded by the use of the Bessel function equality

$$e^{\lambda \cos \Omega (\tau+\tau')} = \sum_{n=-\infty}^{\infty} I_n(\lambda)\, e^{in\Omega(\tau+\tau')} \qquad \ldots(2.56)$$

where $I_n(\lambda) = i^{-n} J_n(i\lambda)$. In order that terms with a common exponential dependence may be collected so that the integrations over τ, τ', and v_z may be performed, the coefficients appearing in equation (2.68), namely a, a^2, ... must be evaluated. This is done below.

$$a\frac{KT}{m} = i\gamma\left(e^{i\Omega\tau''} - e^{-i\Omega\tau''}\right)$$

$$a^2\left(\frac{KT}{m}\right)^2 = -\gamma^2\left(e^{i2\Omega\tau''} - 2 + e^{-2i\Omega\tau''}\right)$$

$$a^3\left(\frac{KT}{m}\right)^3 = -i\gamma^3\left(e^{i3\Omega\tau''} - 3e^{i\Omega\tau''} + 3e^{-i\Omega\tau''} - e^{-i3\Omega\tau''}\right)$$

$$b\left(\frac{KT}{m}\right) = \varepsilon\gamma\left(2 - e^{i\Omega\tau''} - e^{-i\Omega\tau''}\right)$$

$$b^2\left(\frac{KT}{m}\right)^2 = \gamma^2\left(6 - 4e^{i\Omega\tau''} - 4e^{-i\Omega\tau''} + e^{i2\Omega\tau''} + e^{-i2\Omega\tau''}\right)$$

$$b^3\left(\frac{KT}{m}\right)^3 = \varepsilon\gamma^3\Big(20 - 15\left(e^{i\Omega\tau''} + e^{-i\Omega\tau''}\right) + 6\left(e^{i2\Omega\tau''} + e^{-i2\Omega\tau''}\right) - \left(e^{i3\Omega\tau''} + e^{-i3\Omega\tau''}\right)\Big) \qquad \ldots(2.57)$$

$$ab\left(\frac{KT}{m}\right)^2 = i\varepsilon\gamma^2\Big(2\left(e^{i\Omega\tau''} - e^{-i\Omega\tau''}\right) - e^{i2\Omega\tau''} + e^{-i2\Omega\tau''}\Big)$$

$$a^2b\left(\frac{KT}{m}\right)^3 = -\varepsilon\gamma^3\Big(-4 + e^{i\Omega\tau''} + e^{-i\Omega\tau''} + 2\left(e^{i2\Omega\tau''} + e^{-i2\Omega\tau''}\right) - e^{i3\Omega\tau''} - e^{-i3\Omega\tau''}\Big)$$

$$ab^2\left(\frac{KT}{m}\right)^3 = i\gamma^3\Big(5\left(e^{i\Omega\tau''} - e^{-i\Omega\tau''}\right) - 4\left(e^{i2\Omega\tau''} - e^{-i2\Omega\tau''}\right) + e^{i3\Omega\tau''} - e^{-i3\Omega\tau''}\Big)$$

where $\quad \gamma = \dfrac{K_x}{2\Omega}\,\dfrac{KT}{m}$

$$\tau'' = \tau + \tau'$$

In the next section, the integration with respect to τ' in equation (2.55) will be performed. This requires that the terms with a common exponential dependence in τ' be collected. After some manipulation, equation (2.55) may be expressed as follows:

$$
\begin{aligned}
\langle f_{32}\rangle_{\perp} = {} & \frac{z^2 e^2}{mKT} E_{1x} E_{2x}^{*} e^{i(\underline{K}\cdot\underline{r} - \omega t)} e^{-\lambda} \sum_{n=-\infty}^{\infty} \int_0^{\infty}\!\!\int_0^{\infty} d\tau\, d\tau'\, g \exp i\Big[n\Omega(\tau+\tau') \\
& + (\omega - K_z V_z)\tau + (\omega_1 - K_{1z} V_z)\tau' \Big] \Big\{ e^{-i\varepsilon\Omega\tau} \Big[\rho_1 \Big[2 I_{n-\varepsilon} - \frac{\delta_1^+}{2} \big((\varepsilon+1) I_{n-1} \\
& + (1-\varepsilon) I_{n+1} \big) + \frac{\lambda}{4} \Big(8 I_{n-\varepsilon} + 4 I_{n-(1+\varepsilon)} + 4 I_{n+1-\varepsilon} + \frac{\delta_1^+}{2} \Big(8 I_{n-1} + 8 I_{n+1} \\
& + 8\varepsilon \frac{n}{\lambda} I_n - 2(\varepsilon+1) I_{n-2} + 2(\varepsilon-1) I_{n+2} - 12 I_n \Big)\Big)\Big] - \frac{\lambda}{2}\Big(1 - \frac{K_2 V_z}{\omega_2}\Big) \Big[(\varepsilon-1) I_{n-(1+2\varepsilon)} \\
& + (\varepsilon+1) I_{n+1-2\varepsilon} + 2 I_{n-2\varepsilon} - \frac{\delta_1^+}{2} \Big(-4 I_{n-\varepsilon} + 2 I_{n-(1+\varepsilon)} + 2 I_{n+1-\varepsilon} \Big) \Big] \\
& - \gamma \Big(\rho_1 V_z \frac{m}{KT} - \frac{K_2}{\omega_2} \Big) \Big(\frac{E_{1z}}{E_{1x}} \Big) \Big((1+\varepsilon) I_{n-1} + (\varepsilon-1) I_{n+1} - 2\varepsilon I_n \Big) + \Big(1 - \frac{K_2 V_z}{\omega_2} \Big) \Big(-2 \\
& + \delta_1^+ + \varepsilon V_z \frac{K_x}{\Omega} \Big(\frac{E_{1z}}{E_{1x}} \Big) I_{n-\varepsilon} \Big] + \frac{\lambda}{2} \Big(1 - \frac{K_2 V_z}{\omega_2} \Big) \Big[(\varepsilon-1) I_{n-(1+\varepsilon)} + (\varepsilon+1) I_{n+(1+\varepsilon)} \\
& + 2 I_{n-\varepsilon} - \frac{\delta_1^+}{2} \big(-4 I_n + 2 I_{n-1} + 2 I_{n+1} \big) - \varepsilon \Big(1 - \frac{K_2 V_z}{\omega_2} \Big) V_z \frac{K_x}{\Omega} \Big(\frac{E_{1z}}{E_{1x}} \Big) I_n \Big\}
\end{aligned}
$$

...(2.58)

$$\langle v_x f_{32}\rangle_\perp = \frac{Z^2 e^2}{m KT} E_{1x} E_{2x}^* \, e^{i(\underline{K}\cdot\underline{r}-\omega t)} e^{-\lambda} \sum_{n=-\infty}^{\infty} \int_0^\infty \int_0^\infty d\tau\, d\tau'\, g \exp i \Big[n\Omega(\tau+\tau')$$

$$+ (\omega - K_z v_z)\tau + (\omega_1 - K_{1z} v_z)\tau' \Big] \Big\{ e^{-i\varepsilon\Omega\tau} \Big[-\gamma \rho_1 \Big[\frac{8(n-\varepsilon)}{\lambda} I_{n-\varepsilon} - \frac{\delta_1^+}{2}\big(-8\varepsilon I_n$$

$$+ 2(\varepsilon-1) I_{n-1} + 2(\varepsilon+1) I_{n+1} + 2(\varepsilon+1) I_{n-2} - 2(1-\varepsilon) I_{n+2} \big) - \frac{\lambda}{4}\Big(\frac{16(n-\varepsilon)}{\lambda} I_{n-\varepsilon}$$

$$- 4 I_{n-(2+\varepsilon)} + 4 I_{n+2+\varepsilon} - \frac{\delta_1^+}{2}\Big(-8\varepsilon I_n - 20\frac{n}{\lambda} I_n + 2\varepsilon (I_{n-1} + I_{n+1})$$

$$+ 4(2+\varepsilon) I_{n-2} - 4(2-\varepsilon) I_{n+2} - 2(1+\varepsilon) I_{n-3} + 2(1-\varepsilon) I_{n+3} \Big)\Big)\Big]$$

$$+ 2\gamma\Big(1 - \frac{K_z v_z}{\omega_2}\Big)\Big[\varepsilon I_{n-2\varepsilon} - \delta_1^+ \frac{(n-\varepsilon)}{\lambda} I_{n-\varepsilon} + \Big(2 - \delta_1^+ - \varepsilon \frac{K_x}{\Omega} v_z \frac{E_{1z}}{E_{1x}} \Big) \frac{(n-\varepsilon)}{\lambda} I_{n-\varepsilon}$$

$$+ \frac{\lambda}{4}\Big(-2\varepsilon I_{n-2\varepsilon} + 4\frac{(n-2\varepsilon)}{\lambda} I_{n-2\varepsilon} + (\varepsilon-1) I_{n-2(1+\varepsilon)} + (1+\varepsilon) I_{n+2(1-\varepsilon)}$$

$$- \frac{\delta_1^+}{2}\Big(-8\frac{(n-\varepsilon)}{\lambda} I_{n-\varepsilon} + 2 I_{n-(2+\varepsilon)} - 2 I_{n+2-\varepsilon} \Big)\Big)\Big] + \Big(\rho_1 v_z - \frac{KT}{m}\frac{K_z}{\omega_2} \Big) \frac{E_{1z}}{E_{1x}} \Big[I_n$$

$$+ \frac{\lambda}{4}\Big(-2 I_n - 4\varepsilon \frac{n}{\lambda} I_n + (1+\varepsilon) I_{n-2} + (1-\varepsilon) I_{n+2} \Big)\Big]\Big] - 2\gamma\Big(1 - \frac{K_z v_z}{\omega_2}\Big)\Big[\varepsilon I_{n-\varepsilon}$$

$$- \delta_1^+ \frac{n}{\lambda} I_n - \varepsilon v_z \frac{K_x}{\Omega}\frac{E_{1z}}{E_{1x}}\frac{n}{\lambda} I_n + \frac{\lambda}{4}\Big(-2\varepsilon I_{n-\varepsilon} + 4\frac{(n-\varepsilon)}{\lambda} I_{n-\varepsilon}$$

$$+ (\varepsilon-1) I_{n-(2+\varepsilon)} + (\varepsilon+1) I_{n+2-\varepsilon} - \frac{\delta_1^+}{2}\Big(-8\frac{n}{\lambda} I_n + 2 I_{n-2} - 2 I_{n+2} \Big)\Big)\Big]\Big\} \qquad \ldots(2.59)$$

$$\langle v_y f_{32}\rangle_\perp = \frac{i\varepsilon Z^2 e^2}{m KT} E_{1x} E_{2x}^* \, e^{i(\underline{K}\cdot\underline{r}-\omega t)} e^{-\lambda} \sum_{n=-\infty}^{\infty} \int_0^\infty \int_0^\infty d\tau\, d\tau'\, g \exp i \Big[n\Omega(\tau+\tau')$$

$$+ (\omega - K_z v_z)\tau + (\omega_1 - K_{1z})\tau' \Big] \Big\{ e^{-i\varepsilon\Omega\tau} \Big[\gamma \rho_1 \Big[-4 I_{n-(1+\varepsilon)} - 4 I_{n+1-\varepsilon}$$

$$+ 8 I_{n-\varepsilon} - \frac{\delta_1^+}{2}\Big(40 I_n - (30+10\varepsilon) I_{n-1} - (30-10\varepsilon) I_{n+1} + (12+8\varepsilon) I_{n-2}$$

$$+ (12-8\varepsilon) I_{n+2} - 2(1+\varepsilon) I_{n-3} - 2(1-\varepsilon) I_{n+3} \Big)\Big] + 2\gamma\Big(1 - \frac{K_z v_z}{\omega_2}\Big)\Big[I_{n-2\varepsilon}$$

$$- \frac{\delta_1^+}{2}\Big(I_{n-(1+\varepsilon)} + I_{n+1-\varepsilon} - \frac{\lambda}{4}\Big(-(4-2\varepsilon) I_{n-(1+2\varepsilon)} - (4+2\varepsilon) I_{n+1-2\varepsilon}$$

$$+ (1-\varepsilon) I_{n-2(1+\varepsilon)} + (1+\varepsilon) I_{n+2(1-\varepsilon)} + 6 I_{n-2\varepsilon} + \frac{\delta_1^+}{2}\Big(12 I_{n-\varepsilon} - 8 I_{n-(1+\varepsilon)}$$

$$-8 I_{n+1+\varepsilon} + 2 I_{n-(2+\varepsilon)} + 2 I_{n+2-\varepsilon}\Big)\Big)\Big] - \Big(\varrho_1 v_z - \frac{KT}{m}\frac{K_2}{\omega_2}\Big)\Big(\frac{E_{1z}}{E_{1x}}\Big)\Big(\varepsilon I_n - \frac{\lambda}{4}\Big(6\varepsilon I_n$$

$$-2(1+\varepsilon) I_{n-1} + 2(1-\varepsilon) I_{n+1} + (\varepsilon+1) I_{n-2} - (1-\varepsilon) I_{n+2}\Big) + \gamma\Big(1 - \frac{K_2 v_z}{\omega_2}\Big)\Big(-2 + \delta_1^+$$

$$+ \varepsilon v_z \frac{K_x}{\Omega}\frac{E_{1z}}{E_{1x}}\Big)\Big(2 I_{n-\varepsilon} - I_{n-(1+\varepsilon)} - I_{n+1-\varepsilon}\Big) - 2\gamma\Big(1 - \frac{K_2 v_z}{\omega_2}\Big)\Big[I_{n-\varepsilon} - \frac{\delta_1^+}{2}\Big(I_{n-1}$$

$$+ I_{n+1}\Big) - \frac{\lambda}{4}\Big(6 I_{n-\varepsilon} - (4-2\varepsilon) I_{n-(1+\varepsilon)} - (4+2\varepsilon) I_{n+1-\varepsilon} + (1-\varepsilon) I_{n-(2+\varepsilon)}$$

$$+ (1+\varepsilon) I_{n+2-\varepsilon} - \frac{\delta_1^+}{2}\Big(-12 I_n + 8 I_{n-1} + 8 I_{n+1} - 2 I_{n-2} - 2 I_{n+2}\Big)\Big)\Big]$$

$$-\varepsilon\gamma\Big(1 - \frac{K_2 v_z}{\omega_2}\Big) v_z \frac{K_x}{\Omega}\frac{E_{1z}}{E_{1x}}\Big(2 I_n - I_{n-1} - I_{n+1}\Big)\Big\} \qquad \ldots(2.60)$$

The integration with respect to τ' in equations (2.58) to (2.60) will now be performed. The asymptotic form of the assumed solutions for the incident fields is as follows:

$$\underline{E}_{1(2)}(\underline{r}', t') = \underline{E}_{1(2)}\, e^{i(\underline{K}_{1(2)}\cdot\underline{r}' - \omega_{1(2)} t')}$$

However, in the limit that $t' \to \infty$, the field must vanish since it must be switched on at some finite time. Therefore, ω_1 may be thought of as having a very small, but finite positive imaginary component. Solutions for steady state and damped oscillations may be obtained from the sloutions for which $Im(\omega_1) > 0$, by an analytic continuation in the frequency domain (see Stix[10], pages 169 and 179). Two types of integrals involving τ' occur in equations (2.58), (2.59) and (2.60).

(i) Integrals with τ' in an exponential term only. These are integrals of the type

$$\int_0^\infty d\tau' \,(\quad)\, \exp i(n\Omega + \omega_1 - K_{1z}V_z)\tau' \qquad \text{where } (\quad) \text{ is independent of } \tau'$$

If ω_1 has a small positive imaginary component, this integral may be shown to be equal to

$$i(\qquad)/(n\Omega + \omega_1 - K_{1z}V_z)$$

(ii) Integrals involving τ' in an exponential term and linearly in the remainder of the integrand. These are integrals of the type

$$\int_0^\infty d\tau' \,(\quad)\, \tau' \exp i(n\Omega + \omega_1 - K_{1z}V_z)\tau'$$

The above integral may easily be integrated by parts, and if ω_1 is taken to have a positive imaginary component, the application of L'Hospital's rule will give the following result:

$$\int_0^\infty d\tau' \,(\quad)\, \tau' \exp i(n\Omega + \omega_1 - K_{1z}V_z)\tau' = -(\qquad)/(n\Omega + \omega_1 - K_{1z}V_z)^2$$

The results obtained in parts (i) and (ii) may be expanded asymptotically as follows:

$$\left.\begin{aligned} \frac{1}{(\omega + n\Omega - K_{1z}V_z)} &\sim \frac{1}{(\omega_1 + n\Omega)}\left[1 + \frac{K_{1z}V_z}{(\omega_1 + n\Omega)} + \ldots\right] \\ \frac{1}{(\omega + n\Omega - K_{1z}V_z)} &\sim \frac{1}{(\omega_1 + n\Omega)^2}\left[1 + \frac{2K_{1z}V_z}{(\omega_1 + n\Omega)} + \ldots\right] \end{aligned}\right\} \qquad \ldots(2.61)$$

The expansions given by equation (2.61) will converge if ω_1 is sufficiently removed from the ion and electron cyclotron frequencies and their harmonics. The integration with respect to τ' in equations (2.58) to (2.60) will now be performed and terms with the exponent $i(\omega + n\Omega - K_z v_z)$ summed. Terms of the order $(v_\theta / v_{p1(2)z})^2$ will be neglected, where v_θ is the thermal velocity for the species being considered, and v_{p1z} and v_{p2} are the phase velocities along the static magnetic field for the first and second incident waves respectively in the plasma. Through the use of the following definitions:

$$\begin{aligned}
\varsigma^{\circ}_{1,n} &= \frac{i}{(\omega_1 + n\Omega)}\left[1 - \frac{K_2 K_{1z}}{\omega_2}\frac{KT}{m}\frac{1}{(\omega_1 + n\Omega)}\right] \\
\varsigma^{'}_{1,n} &= \frac{i K_{1z}}{(\omega_1 + n\Omega)^2} \\
\gamma_{1,n} &= \frac{i\gamma}{(\omega_1 + n\Omega)} \\
\alpha^{\circ}_{1,n} &= \frac{i\varepsilon\Omega}{(\omega_1 + n\Omega)(\omega_1 + (n+\varepsilon)\Omega)} \\
\alpha^{'}_{1,n} &= i K_{1z}\left\{\frac{1}{(\omega_1 + n\Omega)^2} - \frac{1}{(\omega_1 + (n+\varepsilon)\Omega)^2}\right\}
\end{aligned} \qquad \ldots(2.62)$$

equations (2.58) to (2.60) may be rewritten as follows after the integration with respect to τ' is performed:

$$\langle f_{32}\rangle = \frac{Z^2 e^2}{mKT} E_{1x} E_{2x}^{*} e^{i(\underline{K}\cdot\underline{r} - \omega t)} e^{-\lambda} \sum_{n=-\infty}^{\infty} \int_0^\infty d\tau\, g(v_z) \exp i\left[\omega + n\Omega - K_z v_z\right]\tau$$

$$\Big\{ (\varsigma^{\circ}_{1,n+\varepsilon} + \varsigma^{1}_{1,n+\varepsilon})\Big[2I_n - \frac{\delta_1^{+}}{2}\Big((\varepsilon+1) I_{n+\varepsilon-1} + (1-\varepsilon) I_{n+1+\varepsilon}\Big) + \frac{\lambda}{4}\Big(8I_n$$

$$-4I_{n-1} + 4I_{n+1} - \frac{\delta_1^{+}}{2}\Big(-8I_{n+\varepsilon-1} - 8I_{n+\varepsilon-1} - \frac{8\varepsilon(n+\varepsilon)}{\lambda} I_{n+\varepsilon} + 2(\varepsilon+1) I_{n+\varepsilon-2}$$

$$-2(\varepsilon-1)\, I_{n+\varepsilon-2} + 12\, I_{n+\varepsilon}))] + \frac{\lambda}{2}\Big[\alpha^0_{1,n}\Big(1-\frac{K_2V_z}{\omega_2}\Big)+\alpha^1_{1,n}V_z\Big]\Big[(\varepsilon-1)I_{n-(1+\varepsilon)}$$

$$-(\varepsilon+1)I_{n+1-\varepsilon} + 2I_{n-\varepsilon} - \frac{\delta_1^+}{2}(-4I_n + 2I_{n-1}+2I_{n+1})\Big] - \Big[\frac{K_x}{2\Omega}\Big(\rho^0_{1,n+\varepsilon}V_z$$

$$+\rho^1_{1,n+\varepsilon}V_z^2\Big) - \gamma_{1,n+\varepsilon}\frac{K_2}{\omega_2}\Big(1-\frac{K_{1z}V_z}{(\omega_1+(n+\varepsilon)\Omega)}\Big)\Big]\Big[(1+\varepsilon)I_{n-1+\varepsilon}+(\varepsilon-1)I_{n+1+\varepsilon}$$

$$-2\varepsilon I_{n+\varepsilon}\Big]\frac{E_{1z}}{E_{1x}} - \varepsilon\frac{K_x}{\Omega}\frac{E_{1z}}{E_{1x}}\Big[\alpha^0_{1,n}\Big(V_z-\frac{K_2V_z^2}{\omega_2}\Big)+\alpha^1_{1,n}V_z^2\Big]I_n$$

$$+\Big[\frac{\gamma_{1,n+\varepsilon}}{\gamma}\Big(1-\frac{K_2V_z}{\omega_2}\Big)+\rho^1_{1,n+\varepsilon}V_z\Big](-2+\delta_1^+)I_n\Big\} \qquad \ldots(2.63)$$

$$\langle v_x f_{32}\rangle_\perp = \frac{z^2e^2}{mKT}E_{1x}E_{2x}^{*}\, e^{i(\underline{k}\cdot\underline{r}-\omega t)}\, e^{-\lambda}\sum_{n=-\infty}^{\infty}\int_0^\infty d\tau\, g(v_z)\exp i\big[\omega+n\Omega-K_zV_z\big]\tau$$

$$\gamma\Big\{-\Big(\rho^0_{1,n+\varepsilon}+\rho^1_{1,n+\varepsilon}V_z\Big)\Big[\frac{8n}{\lambda}I_n - \frac{\delta_1^+}{2}\Big(-8\varepsilon I_{n+\varepsilon}+2(\varepsilon-1)I_{n+\varepsilon-1}+2(\varepsilon+1)I_{n+\varepsilon+1}$$

$$+2(\varepsilon+1)I_{n-2+\varepsilon}-2(1-\varepsilon)I_{n+2+\varepsilon} - \frac{\lambda}{4}\Big(\frac{16n}{\lambda}I_n - 4I_{n-2}+4I_{n+2}-\frac{\delta_1^+}{2}\Big(-8\varepsilon I_{n+\varepsilon}$$

$$-20\frac{(n+\varepsilon)}{\lambda}I_{n+\varepsilon}+2\varepsilon(I_{n+\varepsilon-1}+I_{n+\varepsilon+1})+4(2+\varepsilon)I_{n+\varepsilon-2}-4(2-\varepsilon)I_{n+\varepsilon-2}$$

$$-2(1+\varepsilon)I_{n+\varepsilon-3}+2(1-\varepsilon)I_{n+3+\varepsilon}\Big)\Big)\Big] - 2\Big[\alpha^0_{1,n}\Big(1-\frac{K_2V_z}{\omega_2}\Big)+\alpha^1_{1,n}V_z\Big]\Big[\varepsilon I_{n-\varepsilon}$$

$$-\delta_1^+\frac{n}{\lambda}I_n + \frac{\lambda}{4}\Big(-2\varepsilon I_{n-\varepsilon}+4\frac{(n-\varepsilon)}{\lambda}I_{n-\varepsilon}+(\varepsilon-1)I_{n-(2+\varepsilon)}+(\varepsilon+1)I_{n+2-\varepsilon}$$

$$-\frac{\delta_1^+}{2}\Big(-\frac{8n}{\lambda}I_n+2I_{n-2}-2I_{n+2}\Big)\Big)\Big] + 2\varepsilon\Big[\alpha^0_{1,n}\Big(V_z-\frac{K_2V_z^2}{\omega_2}\Big)$$

$$+\alpha^1_{1,n}V_z^2\Big]\frac{K_x}{\Omega}\frac{E_{1z}}{E_{1x}}\frac{n}{\lambda}I_n + (4-2\delta_1^+)\Big[\frac{\gamma_{1,n+\varepsilon}}{\gamma}\Big(1-\frac{K_2V_z}{\omega_2}\Big)+\rho^1_{1,n+\varepsilon}V_z\Big]\frac{n}{\lambda}I_n$$

$$+\frac{1}{\gamma}\Big[\rho^0_{1,n+\varepsilon}V_z+\rho^1_{1,n+\varepsilon}V_z^2-\frac{KT}{m}\frac{K_2}{\omega_2}\frac{\gamma_{1,n+\varepsilon}}{\gamma}\Big]\frac{E_{1z}}{E_{1x}}\Big[I_{n+\varepsilon}+\frac{\lambda}{4}\Big(-2I_{n+\varepsilon}$$

$$-4\varepsilon\frac{(n+\varepsilon)}{\lambda}I_{n+\varepsilon}+(1+\varepsilon)I_{n+\varepsilon+2}+(1-\varepsilon)I_{n+\varepsilon-2}\Big)\Big]\Big\} \qquad \ldots(2.64)$$

$$\langle v_y f_{32}\rangle_\perp = \frac{i\varepsilon Z^2 e^2}{mKT} E_{1x} E_{2x}^* e^{i(\underline{K}\cdot\underline{r}-\omega t)} e^{-\lambda} \sum_{n=-\infty}^{\infty} \int_0^{\infty} d\tau\, g(v_z) \exp i\left[\omega + n\Omega - K_z v_z\right]$$

$$\gamma \Big\{ \left(\rho^{0}_{1,n+\varepsilon} + \rho^{1}_{1,n+\varepsilon} v_z \right) \Big[-4I_{n-1} - 4I_{n+1} + 8I_n - \frac{\delta_1^+}{2} \Big((6+2\varepsilon) I_{n-1+\varepsilon}$$

$$+ (6-2\varepsilon) I_{n+1+\varepsilon} - 2(\varepsilon+1) I_{n-2+\varepsilon} + 2(\varepsilon-1) I_{n+2+\varepsilon} - 8I_{n+\varepsilon} \Big) + \frac{\lambda}{4} \Big(-24 I_n$$

$$+ 16 I_{n-1} + 16 I_{n+1} - 4I_{n-2} - 4I_{n+2} - \frac{\delta_1^+}{2} \Big(40 I_{n+\varepsilon} - (30+10\varepsilon) I_{n+\varepsilon-1}$$

$$-(30-10\varepsilon) I_{n+\varepsilon+1} + (12+8\varepsilon) I_{n-2+\varepsilon} + (12-8\varepsilon) I_{n+2+\varepsilon} - 2(1+\varepsilon) I_{n-3+\varepsilon}$$

$$-2(1-\varepsilon) I_{n+3+\varepsilon} \Big)\Big)\Big] - 2\Big[\alpha^{0}_{1,n} \left(1 - \frac{K_z v_z}{\omega_2}\right) + \alpha^{1}_{1,n} v_z \Big] \Big[I_{n-\varepsilon} - \frac{\delta_1^+}{2} \big(I_{n-1}$$

$$+ I_{n+1} \big) - \frac{\lambda}{4} \Big(6 I_{n-\varepsilon} - (4-2\varepsilon) I_{n-(1+\varepsilon)} - (4+2\varepsilon) I_{n+1-\varepsilon} + (1-\varepsilon) I_{n-(2+\varepsilon)}$$

$$+ (1-\varepsilon) I_{n+2-\varepsilon} - \frac{\delta_1^+}{2} \big(-12 I_n + 8 I_{n-1} + 8 I_{n+1} - 2I_{n-2} - 2I_{n+2} \big)\Big)\Big]$$

$$- \varepsilon \frac{K_x}{\Omega} \frac{E_{1z}}{E_{1x}} \Big[\alpha^{0}_{1,n} \left(v_z - \frac{K_z v_z^2}{\omega_2} \right) + \alpha^{1}_{1,n} v_z^2 \Big] \Big[2I_n - I_{n-1} - I_{n+1} \Big] - (2 - \delta_1^+) \Big[$$

$$\rho^{0}_{1,n+\varepsilon} v_z + \rho^{1}_{1,n+\varepsilon} v_z^2 - \frac{KT}{m} \frac{K_z}{\omega_2} \frac{\gamma_{1,n+\varepsilon}}{\gamma} \Big] \frac{E_{1z}}{E_{1x}} \Big[\varepsilon I_{n+\varepsilon} - \frac{\lambda}{4} \Big(6\varepsilon I_{n+\varepsilon}$$

$$-2(1+2\varepsilon) I_{n+\varepsilon-1} + 2(1-2\varepsilon) I_{n+\varepsilon+1} + (\varepsilon+1) I_{n+\varepsilon-2} - (1-\varepsilon) I_{n+2+\varepsilon} \Big)\Big] \Big\}$$

...(2.65)

The integrals still to be evaluated are of the type

$$F_p = \frac{K_z}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv_z \int_0^{\infty} d\tau\, v_z^p \exp\left(- \frac{m v_z^2}{2KT} + i\tau(\omega + n\Omega - K_z v_z) \right) \qquad \text{...(2.66)}$$

where p = 0, 1, 2, or 3 . The results given in Stix[10] for integrals of this type may be generalized to include the case where p = 3. In order to demonstrate the convergence of the integration with respect to τ in equation (2.66) for either positive or negative values for the

imaginary component of ω, the square in the exponent may be completed to give[10]

$$F_p = \frac{K_z}{\sqrt{\pi}} \int_0^{\infty} d\tau \, \exp\left[i(\omega + n\Omega)\tau - K_z^2 \frac{KT}{m} \tau^2 \right] \int_{-\infty}^{\infty} dv_z \, v_z^p \exp -\frac{m}{2KT}\left[v_z + i K_z \frac{KT}{m}\tau \right]^2 \quad \ldots(2.67)$$

As is evident from equation (2.67), convergence will exist for either positive or negative values for Im(ω). Therefore, F_p may be considered to be an entire function of ω.

In the next step, F_1, F_2, and F_3 will be expressed in terms of F_o. For Im(ω) > 0, the integration with respect to τ in equation (2.66) may be performed first, to give

$$F_p = \frac{iK_z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dv_z \, v_z^p \, e^{-\frac{mv_z^2}{2KT}}}{(\omega + n\Omega - K_z v_z)} \quad \ldots(2.68)$$

Since F_p is an entire function of w (see equation (2.67)), the value for F_p in the lower ω half-plane may be obtained by analytic continuation.

From equation (2.68)

$$F_1 = \frac{iK_z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dv_z \, v_z \, e^{-\frac{mv_z^2}{2KT}}}{(\omega + n\Omega - K_z v_z)}$$

The integrand in the above equation may be expressed as follows:

$$\frac{v_z}{(\omega + n\Omega - K_z v_z)} = \frac{-1}{K_z} + \frac{(\omega + n\Omega)}{K_z} \frac{1}{(\omega + n\Omega - K_z v_z)}$$

The integration with respect to v_z may then be performed to give

$$F_1 = -i\sqrt{\frac{2KT}{m}} + \frac{(\omega + n\Omega)}{K_z} F_o \qquad \ldots(2.69)$$

In a similar fashion

$$F_2 = -i\frac{(\omega + n\Omega)}{K_z}\sqrt{\frac{2KT}{m}} + \left(\frac{\omega + n\Omega}{K_z}\right)^2 F_o$$

$$F_3 = -\frac{i}{2}\left(\frac{2KT}{m}\right)^{3/2} - i\left(\frac{\omega + n\Omega}{K_z}\right)^2\sqrt{\frac{2KT}{m}} + \left(\frac{\omega + n\Omega}{K_z}\right)^3 F_o \qquad \ldots(2.70)$$

Either an asymptotic or convergent series may be written for F_o, the latter being required when ω is close to a multiple of the respective gyro-frequency or when the Landau damping of a wave is significant. The convergent series for F_o is obtained by first integrating with respect to v_z in the expression given by equation (2.67) for F_o. By defining

$$\alpha_n = \frac{(\omega + n\Omega)}{K_z}\sqrt{\frac{m}{2KT}} \qquad \ldots(2.71)$$

and making the following transformation of variables,

$$u = K_z\tau - i\sqrt{\frac{2m}{KT}}\,\alpha_n$$

the integration with respect to τ may be performed. This gives

$$F_o = \frac{K_z}{|K_z|}\sqrt{\pi}\, e^{-\alpha_n^2} + 2i\, S(\alpha_n) \qquad \ldots(2.72)$$

where S(z) is the complex error function and is given by

$$S(z) = e^{-z^2} \int_0^z e^{t^2} dt \qquad \ldots(2.73)$$

The function F_o at this stage may be evaluated in two ways. One is the use of tabulated values given by B.D. Fried and S.D.Conte[24] for the function

$$\omega(z) = \pi^{-1/2} \left\{ \sqrt{\pi} \exp -(z^2) + 2i\, S(z) \right\}$$

Another is the development of either a convergent or asymptotic series, depending upon the absolute value of α_n.

A convergent series is obtained by the integration of equation (2.73) by parts. This yields

$$S(z) = \left\{ e^{-z^2} \left[t e^{t^2} \right]_0^z - \int_0^z 2t^2 e^{t^2} dt \right\}$$

$$= z - e^{-z^2} \int_0^z 2t^2 e^{t^2} dt$$

This procedure may be repeated to give the series

$$S(z) = z - \frac{2z^3}{3.1} + \frac{2.2\, z^5}{5.3.1} \qquad \ldots(2.74)$$

Convergence will exist for $|z| < 1$, which implies that if the convergent series is to be used, $|\alpha_n| < 1$.

An asymptotic expansion for F_o, valid when $|\alpha_n| > 1$, is given in Stix[10] (pp.179). The technique used in obtaining this expansion will

be briefly outlined below. From equation (2.68)

$$F_o = \frac{-i K_z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dv_z\, e^{-\frac{mv_z^2}{2KT}}}{K_z\left(v_z - \left(\frac{\omega+n\Omega}{K_z}\right)\right)} = \frac{-i}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}\, dt}{t-\alpha_n}$$

$$= \frac{-i}{\sqrt{\pi}}\, P\int_{-\infty}^{\infty} \frac{e^{-t^2}\, dt}{t-\alpha_n} + \sqrt{\pi}\, \frac{K_z}{|K_z|} \exp(-\alpha_n^2) \qquad \ldots(2.75)$$

where the symbol P denotes the principal value of the integral. If the expression for F_o in equation (2.69) is compared with equation (2.75), an alternate expression for S(z) is obtained, namely

$$S(z) = \frac{1}{2\sqrt{\pi}}\, P\int_{-\infty}^{\infty} \frac{e^{-t^2}\, dt}{t+z} \qquad \ldots(2.76)$$

The asymptotic expansion for S(z) may be obtained by first expanding the integral given in equation (2.76) for z real, then forming the analytic continuation into the complex plane using a Taylor series. The asymptotic expansion for S(z) in the region where $|Re(\alpha_n)| > |Im(\alpha_n)|$ is given by

$$S(z) = \frac{1}{2z} + \frac{1}{2.2\, z^3} + \frac{1 \cdot 3}{2 \cdot 2 \cdot 2\, z^5} + \ldots \qquad \ldots(2.77)$$

In the region where $|Re(\alpha_n)| < |Im(\alpha_n)|$, a different expansion is required. Thus the asymptotic expansion for S(z) exhibits a Stokes phenomenon (see Stix[10] (pp 180) and Morse and Feshbach[25]). An asymptotic expansion in this region will not be required in this thesis.

Equations (2.63) to (2.65) will now be used for obtaining expressions for the electron and ion driving current. The remaining integra-

tions with respect to τ and v_z in these equations may be evaluated in terms of F_o, F_1, F_2, and F_3. F_o will be approximated as follows:

$$F_o = \sqrt{\pi}\frac{K_z}{|K_z|} + 2i\,\alpha_n \quad \text{for } |\alpha_n| << 1 \quad (\text{convergent series})$$

$$F_o = \sqrt{\pi}\frac{K_z}{|K_z|}\exp -(\alpha_n)^2 + \frac{i}{\alpha_n}\left(1 + \frac{1}{2\alpha_n^2} + \cdots\right) \quad \text{for } |\alpha_n| >> 1 \quad (\text{asymptotic series}) \qquad \ldots(2.78)$$

Where α_n is given by equation (2.71). In the second expression in equation (2.74), the first term gives the contribution of the collisionless damping effects, the first component in the second term gives the normal cold plasma contribution, while the second component gives the first temperature correction term.

In obtaining the various current components, only terms to the zero'th order in λ (where λ is given by equation (2.53)) will be included. Such an analysis will be sufficiently general to account for Landau or Cherenkov damping and cyclotron damping at the cyclotron frequency for the species. Expansions to higher orders in λ may be used to account for finite Larmor radius effects as well as transit time damping, that is, the effects of fluctuations in the static magnetic field. Such a generalization would also account for the effects of cyclotron damping at the higher harmonics of the gyrofrequency.

From equation (2.39) it is evident that the second order current, given by the velocity moments of $f_3^{\pm}$ will have three separate components, two driving components and one induced component. The notation that will be used to represent the driving terms is as follows:

$$J_{3d}^{\pm(K)} = n_o e \mathcal{E}_{\pm} \langle \underline{v}\, f_{3K}^{\pm} \rangle = \mathcal{E}_{\pm} D^{\pm} \sum_{n=-\infty}^{\infty} \omega\, \Pi_{i(n)}^{\pm(K)} \qquad \ldots(2.79)$$

where i = 1,2,3 and denotes components in the x, y, and z directions

respectively.

k = 1, 2 and complies with the notation used in equation (2.39).

n refers to α_n used in the evaluation of F_o, F_1, F_2, and F_3.

$$D^- = \left(\frac{e}{m_-}\right) \frac{\omega_{p-}^2}{\omega\Omega_-} \frac{K_x}{2K_z} \sqrt{\frac{m_-}{2KT^-}}\, E_{1x} E_{2x}^*$$

$$D^+ = \left(\frac{e}{m_+}\right) \frac{\omega_{p+}^2}{\omega\Omega_+} \frac{K_x}{2K_z} \sqrt{\frac{m_+}{2KT^+}}\, E_{1x} E_{2x}^* = \sqrt{\frac{m_-}{m_+}}\, D^- \quad (\text{for } T^+ = T^-)$$

The Π coefficients are evaluated to order zero in $\lambda^{\pm}$ through the use of equations (2.63) to (2.65), and are enumerated below for k = 2. Only the non-zero terms are listed.

$$\Pi_{3(0)}^{+(2)} = \frac{1}{\gamma}\Big\{ (2-\delta_1^+)\left(\rho_{1,1}^{0} F_1 + \rho_{1,1}^{1} F_2\right) - 2\Big(\frac{K_x}{2\Omega}\left(\rho_{1,1}^{0} F_2 + \rho_{1,1}^{1} F_3\right)$$

$$- \gamma_{1,1} \frac{K_2}{\omega_2}\left(F_1 - \frac{K_{1z}}{(\omega_1+\Omega_+)} F_2\right)\Big) \frac{E_{1z}}{E_{1x}} - \frac{K_x}{\Omega_+} \frac{E_{1x}}{E_{1z}}\left[\alpha_{1,0}\left(F_2 - \frac{K_2 F_3}{\omega_2}\right) + \alpha_{1,0}^{1} F_3\right]$$

$$- (2-\delta_1^+)\left(\frac{\gamma_{1,1}}{\gamma}\left(F_1 - \frac{K_2 F_2}{\omega_2}\right) + \rho_{1,1}^{1} F_2\right)\Big\}$$

$$\Pi_{3(-1)}^{+(2)} = \frac{2}{\gamma}\left\{ \frac{K_x}{2\Omega_+}\left(\rho_{1,0}^{0} F_2 + \rho_{1,0}^{1} F_3\right) - \gamma_{1,0}\frac{K_2}{\omega_2}\left(F_1 - \frac{K_{1z} F_2}{\omega_1}\right)\right\} \frac{E_{1z}}{E_{1x}}$$

$$+ \frac{\lambda}{\gamma}\left\{ \left(\rho_{1,0}^{0} + \rho_{1,0}^{1} F_2\right)\left(1 - \frac{3\delta_1^+}{4}\right) - \left(\alpha_{1,-1}^{0}\left(F_1 - \frac{K_2}{\omega_2} F_2\right) + \alpha_{1,-1}^{1} F_2\right)\frac{\delta_1^+}{2}\right\}$$

$$\Pi_{3(0)}^{-(2)} = \frac{1}{\gamma}\Big\{ (2-\delta_1^+)\left(\rho_{1-1}^{0} F_1 + \rho_{1,-1}^{1} F_2\right) + 2\Big(\frac{K_x}{2\Omega_-}\left(\rho_{1,-1}^{0} F_2 + \rho_{1,-1}^{1} F_3\right)$$

$$- \gamma_{1,-1}\frac{K_2}{\omega_2}\left(F_1 - \frac{K_{1z} F_2}{\omega_1 - \Omega_-}\right)\Big) \frac{E_{1z}}{E_{1x}} + \frac{K_x}{\Omega_-}\frac{E_{1z}}{E_{1x}}\left(\alpha_{1,0}^{0}\left(F_2 - \frac{K_2 F_3}{\omega_2}\right)\right.$$

$$+ \alpha^1_{1,0} F_3 \quad -(2-\delta_1^+)\left(\frac{\gamma_{1,-1}}{\gamma}\left(F_1 - \frac{K_2 F_2}{\omega_2}\right) + \rho^1_{1,1} F_2 \right)\Big\}$$

$$\Pi^{-(2)}_{3(1)} = -\frac{2}{\gamma}\left\{ \frac{K_x}{2\Omega_-}\left(\rho^0_{1,0}F_2 + \rho^1_{1,0}F_3\right) - \gamma_{1,0}\frac{K_2}{\omega_2}\left(F_1 - \frac{K_{1z}}{\omega_1}F_2\right)\right\}\frac{E_{1z}}{E_{1x}}$$

$$\Pi^{+(2)}_{1(1)} = -2(2-\delta_1^+)\left(\rho^0_{1,2} F_0 + \rho^1_{1,2}F_1\right) - (2-\delta_1^+)\left(\alpha^0_{1,1}\left(F_0 - \frac{K_2 F_1}{\omega_2}\right) + \alpha^1_{1,1}F_1\right)$$

$$+ (2-\delta_1^+)\left(\frac{\gamma_{1,2}}{\gamma}\left(F_0 - \frac{K_2 F_1}{\omega_2}\right) + \rho^1_{1,2}F_1\right) + 2\left(\alpha^0_{1,1}\left(F_1 - \frac{K_2 F_2}{\omega_2}\right) + \alpha^1_{1,1}F_2\right)\frac{K_x}{\Omega}\frac{E_{1z}}{E_{1x}}$$

$$\Pi^{+(2)}_{1(-1)} = 4(1-\delta_1^+)\left(\rho^0_{1,0}F_0 + \rho^1_{1,0}F_1\right) - \delta_1^+\left(\alpha^0_{1,-1}\left(F_0 - \frac{K_2F_1}{\omega_2}\right) + \alpha^1_{1,-1}F_1\right)$$

$$-(2-\delta_1^+)\left(\frac{\gamma_{1,0}}{\gamma}\left(F_0 - \frac{K_2F_1}{\omega_2}\right) + \rho^1_{1,0}F_1\right) - \left(\alpha^0_{1,-1}\left(F_1 - \frac{K_2F_2}{\omega_2}\right) + \alpha^1_{1,-1}F_2\right)\frac{K_x}{\Omega}\frac{E_{1z}}{E_{1x}}$$

$$+ \frac{1}{\gamma}\left(\rho^0_{1,0}F_1 + \rho^1_{1,0}F_2 - \frac{KT}{m}\frac{K_2}{\omega_2}\frac{\gamma_{1,0}}{\gamma}F_0\right)\frac{E_{1z}}{E_{1x}}$$

$$\Pi^{+(2)}_{1(-2)} = 2\delta_1^+\left(\rho^0_{1,-1}F_0 + \rho^1_{1,-1}F_1\right)$$

$$\Pi^{-(2)}_{1(1)} = -4(1-\delta_1^+)\left(\rho^0_{1,0}F_0 + \rho^1_{1,0}F_1\right) + \delta_1^+\left(\alpha^0_{1,1}\left(F_0 - \frac{K_2}{\omega_2}F_1\right) + \alpha^1_{1,1}F_1\right)$$

$$+ (2-\delta_1^+)\left(\frac{\gamma_{1,0}}{\gamma}\left(F_0 - \frac{K_2F_1}{\omega_2}\right) + \rho^1_{1,0}F_1\right) - \left(\alpha^0_{1,1}\left(F_1 - \frac{K_2F_2}{\omega_2}\right) + \alpha^1_{1,1}F_2\right)\frac{K_x}{\Omega}\frac{E_{1z}}{E_{1x}}$$

$$+ \frac{1}{\gamma}\left(\rho^0_{1,0}F_1 + \rho^1_{1,0}F_2 - \frac{KT}{m}\frac{K_2}{\omega_2}\frac{\gamma_{1,0}}{\gamma}\right)\frac{E_{1z}}{E_{1x}}$$

$$\Pi^{-(2)}_{1(-1)} = 2(2-\delta_1^+)\left(\rho^1_{1,-2}F_0 + \rho^1_{1,-2}F_1\right) + (2-\delta_1^+)\left(\alpha^0_{1,-1}\left(F_0 - \frac{K_2F_1}{\omega_2}\right) + \alpha^1_{1,-1}F_1\right)$$

$$-(2-\delta_1^+)\left(\frac{\gamma_{1,-2}}{\gamma}\left(F_0 - \frac{K_2F_1}{\omega_2}\right) + \rho^1_{1,-2}F_1\right) + \left(\alpha^0_{1,-1}\left(F_0 - \frac{K_2F_2}{\omega_2}\right) + \alpha^1_{1,-1}F_2\right)\frac{K_x}{\Omega}\frac{E_{1z}}{E_{1x}}$$

$$\Pi^{-(2)}_{1(2)} = -2\delta_1^+\left(\rho^0_{1,1}F_0 + \rho^1_{1,1}F_1\right)$$

$$\Pi^{+(2)}_{2(0)} = i\left\{ 4(2-\delta_1^+)\left(\rho^0_{1,1}F_0 + \rho^1_{1,1}F_1\right) - 2\frac{K_x}{\Omega}\frac{E_{1z}}{E_{1x}}\left(\alpha^0_{1,0}\left(F_1 - \frac{K_2F_2}{\omega_2}\right) + \alpha^1_{1,0}F_2\right)\right.$$

$$-2(2-\delta_1^+)\left(\frac{\gamma_{1,1}}{\gamma}\left(F_1-\frac{K_2F_2}{\omega_2}\right)+\rho^1_{1,1}F_1\right)\Big\}$$

$$\Pi^{+(2)}_{2(1)} = i\Big\{-2(2-\delta_1^+)\left(\rho^0_{1,2}F_0+\rho^1_{1,2}F_1\right)-(2-\delta_1^+)\left(\alpha^0_{1,1}\left(F_1-\frac{K_2F_2}{\omega_2}\right)+\alpha^1_{1,1}F_1\right)$$

$$+(2-\delta_1^+)\left(\frac{\gamma_{1,2}}{\gamma}\left(F_0-\frac{K_2F_1}{\omega_2}\right)+\rho^1_{1,2}F_1\right)+\frac{K_x}{\Omega_+}\frac{E_{1z}}{E_{1x}}\left(\alpha^0_{1,1}\left(F_1-\frac{K_2F_2}{\omega_2}\right)+\alpha^1_{1,1}F_2\right)\Big\}$$

$$\Pi^{+(2)}_{2(-1)} = i\Big\{-4(1-\delta_1^+)\left(\rho^0_{1,0}F_0+\rho^1_{1,0}F_1\right)+\delta_1^+\left(\alpha^0_{1,-1}\left(F_0-\frac{K_2F_1}{\omega_2}\right)+\alpha^1_{1,-1}F_1\right)$$

$$+(2-\delta_1^+)\left(\frac{\gamma_{1,0}}{\gamma}\left(F_0-\frac{K_2F_1}{\omega_2}\right)+\rho^1_{1,0}F_1\right)+\frac{K_x}{\Omega_+}\frac{E_{1z}}{E_{1x}}\left(\alpha^0_{1,-1}\left(F_1-\frac{K_2F_2}{\omega_2}\right)+\alpha^1_{1,-1}F_2\right)$$

$$-\frac{1}{\gamma}\left(\rho^0_{1,0}F_1+\rho^1_{1,0}F_2-\frac{KT}{m}\frac{K_2}{\omega_2}\frac{\gamma_{1,0}}{\gamma}F_0\right)\frac{E_{1z}}{E_{1x}}\Big\}$$

$$\Pi^{+(2)}_{2(-2)} = -2i\,\delta_1^+\left(\rho^0_{1,-1}F_0+\rho^1_{1,-1}F_1\right)$$

$$\Pi^{-(2)}_{2(0)} = -i\Big\{4(2-\delta_1^+)\left(\rho^0_{1,-1}F_0+\rho^1_{1,-1}F_1\right)+\frac{2K_x}{\Omega_-}\frac{E_{1z}}{E_{1x}}\left(\alpha^0_{1,0}\left(F_1-\frac{K_2F_2}{\omega_2}\right)+\alpha^1_{1,0}F_2\right)$$

$$-2(2-\delta_1^+)\left(\frac{\gamma_{1,-1}}{\gamma}\left(F_0-\frac{K_2F_1}{\omega_2}\right)+\rho^1_{1,-1}F_1\right)\Big\}$$

$$\Pi^{-(2)}_{2(-1)} = -i\Big\{-2(2-\delta_1^+)\left(\rho^0_{1,-2}F_1+\rho^1_{1,-2}F_1\right)-(2-\delta_1^+)\left(\alpha^0_{1,-1}\left(F_0-\frac{K_2F_1}{\omega_2}\right)+\alpha^1_{1,-1}F_1\right)$$

$$+(2-\delta_1^+)\left(\frac{\gamma_{1,-2}}{\gamma}\left(F_0-\frac{K_2F_1}{\omega_2}\right)+\rho^1_{1,-2}F_1\right)-\frac{K_x}{\Omega_-}\frac{E_{1z}}{E_{1x}}\left(\alpha^0_{1,-1}\left(F_1-\frac{K_2F_2}{\omega_2}\right)+\alpha^1_{1,-1}F_2\right)\Big\}$$

$$\Pi^{-(2)}_{2(1)} = -i\Big\{-4(1-\delta_1^+)\left(\rho^0_{1,0}F_1+\rho^1_{1,0}F_1\right)+\delta_1^+\left(\alpha^0_{1,1}\left(F_0-\frac{K_2F_1}{\omega_2}\right)+\alpha^1_{1,1}F_1\right)$$

$$+(2-\delta_1^+)\left(\frac{\gamma_{1,0}}{\gamma}\left(F_0-\frac{K_2F_1}{\omega_2}\right)+\rho^1_{1,0}F_1\right)-\frac{K_x}{\Omega_-}\frac{E_{1z}}{E_{1x}}\left(\alpha^0_{1,1}\left(F_1-\frac{K_2F_2}{\omega_2}+\alpha^0_{1,1}F_2\right)\right.$$

$$\left.+\frac{1}{\gamma}\left(\rho^0_{1,0}F_1+\rho^1_{1,1}F_2-\frac{KT}{m}\frac{K_2}{\omega_2}\frac{\gamma_{1,0}}{\gamma}\right)\frac{E_{1z}}{E_{1x}}\right\}$$

$$\Pi^{-(2)}_{2(2)} = 2i\,\delta_1^+\left(\rho^0_{1,1}F_0+\rho^1_{1,1}F_1\right)$$

...(2.80)

2.3.2 Velocity Moments of f_{31}

In this section, the distribution function f_{31} and its velocity moments will be evaluated. From equations (2.38) and (2.39),

$$f_{31}^{\pm} = -\frac{Z_{\pm} e \varepsilon_{\pm}}{m_{\pm}} \int_{-\infty}^{t} dt' \left(\underline{E}_1 + \frac{\underline{v}' \times \underline{B}_1}{c} \right) \cdot \frac{\partial f_2^{\pm *}}{\partial \underline{v}} \qquad \ldots(2.81)$$

In the analysis to follow, specific reference to ions or electrons will be dropped until currents are to be evaluated. The first step in the analysis will be the evaluation of $f_2^*(t')$. This is given by

$$f_2^*(t') = -\frac{Ze\varepsilon}{m} \int_{-\infty}^{t'} \left[\underline{E}_2^*(\underline{r}'', t'') + \frac{\underline{v}'' \times \underline{B}_2^*(\underline{r}'', t'')}{c} \right] \cdot \frac{\partial f_0}{\partial \underline{v}''} \qquad \ldots(2.82)$$

The assumed exponential dependence for the second incident wave is as follows:

$$\underline{E}_2^*(\underline{r}'', t'') = \underline{E}_2^* \, e^{-i(\underline{k}_2 \cdot \underline{r}'' - \omega_2^* t'')}$$

Through the use of the Maxwell equation

$$\nabla \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t}$$

equation (2.82) may be written in terms of $\underline{E}_2^*$ alone. With the aid of equation (2.4) and (2.5), the variables in $\underline{r}''$ and t'' may be expressed in terms of $\underline{r}'$ and t'. If the equilibrium distribution function (f_o) is assumed to be Maxwellian with equal temperatures in the directions perpendicular and parallel to the static magnetic field, and if only the RH circularly polarized portion of the second incident wave is considered in the nonlinear mixing process

$$f_2^*(t') = \frac{Ze\varepsilon}{KT} E_2^* \, e^{-i(K_2 \cdot r' - \omega_2 t')} \int_0^\infty d\tau' \, e^{i(K_2 v_z' - \omega_2^* - \varepsilon\Omega)\tau'}$$

$$\cdot \left[(v_x' - i v_y') f_0 \right] \qquad \ldots(2.83)$$

After the velocity gradient and dot product in equation (2.81) are performed, the variables in $\underline{r}'$ and t' may be expressed in terms of $\underline{r}$ and t by the use of equations (2.4) and (2.5). The justification for this step is discussed in Appendix B. If the propagation vector $\underline{K}_2$ is taken to be parallel to the static magnetic field (see Figure (2.2)), τ' will appear uncoupled from the variable τ in the exponent of the equation for f_{31}. This may be compared with the exponent in equation (2.47) which involves terms such as $\cos \Omega (\tau + \tau')$. This presents a considerable simplification over the analysis for f_{32}, since now the integration with respect to τ' may be performed before any of the velocity integrations or the integration over τ. Two types of integrals are encountered in the integration with respect to τ'. These are

$$I_1 = \int_0^\infty d\tau' \, (\quad) \exp -i(\omega_2^* - K_2 v_z + \varepsilon\Omega)\tau'$$

$$I_2 = \int_0^\infty d\tau' \, (\quad) \tau' \exp -i(\omega_2^* - K_2 v_z + \varepsilon\Omega)\tau' \qquad \ldots(2.84)$$

where ( ) is independent of τ'. The integration may readily be carried out by parts. If ω_2 is assumed to have a positive imaginary component, integrals I_1 and I_2 become

$$I_1 = \frac{-i(\quad)}{(\omega_2 - K_2 v_z + \varepsilon\Omega)} \qquad I_2 = \frac{-(\quad)}{(\omega_2 - K_2 v_z + \varepsilon\Omega)^2}$$

The above results are expanded by the use of an asymptotic expansion as in equation (2.61).

$$I_1 \sim \frac{-i(\quad)}{(\omega_2 + \varepsilon\Omega)}\left[1 + \frac{K_2 v_z}{(\omega_2 + \varepsilon\Omega)} + \ldots\right]$$

$$I_2 \sim \frac{-(\quad)}{(\omega_2 + \varepsilon\Omega)^2}\left[1 + \frac{2K_2 v_z}{(\omega_2 + \varepsilon\Omega)} + \ldots\right] \qquad \ldots(2.85)$$

The expansion will converge if the frequency ω_2 is sufficiently removed from the ion and electron cyclotron frequencies and their harmonics. As in the analysis for the velocity moments of f_{32}, terms of the order of $(v_\theta / v_{plz(2)})^2$ will be neglected, where v_θ is the thermal velocity for the species being considered, and v_{plz} and v_{p2} are the phase velocities along the static magnetic field for the first and second incident waves respectively, in the plasma.

The integrand in the equation for f_{31} will now be expressed in terms of the various powers of the velocity components v_x and v_y (see equation (2.48) for a similar equation for f_{32}).

$$f_{31} = \frac{Z^2 e^2}{mKT} E_{1x} E_{2x}^* e^{i(\underline{K}\cdot\underline{r} - \omega t)} \left(\frac{m}{2\pi KT}\right) \int_0^\infty d\tau\, g(v_z) \exp\Big(iav_x - \frac{mv_x^2}{2KT}$$
$$+ ibv_y - \frac{mv_y^2}{2KT} + i\psi\Big) \left\{\alpha_{11} v_x^2 + \alpha_{12} v_y^2 + i\alpha_{13} v_x v_y + \alpha_{14} v_x + i\alpha_{15} v_y + \alpha_{16}\right\} \qquad \ldots(2.86)$$

where
$$a = -\frac{K_x}{\Omega}\sin\Omega\tau$$

$$b = \frac{\varepsilon K_x}{\Omega}(1-\cos\Omega\tau)$$

$$\psi = (\omega - K_z V_z)\tau$$

$$g(V_z) = \left(\frac{m}{2\pi KT}\right)^{1/2}\left[1 - \frac{K_2 K_{1z}}{\omega_1(\omega_2+\varepsilon\Omega)}\frac{KT}{m}\right]$$

$$\rho_2^1 = \frac{i K_2}{(\omega_2+\varepsilon\Omega)^2}$$

$$\rho_3 = \frac{i K_{1x} K_2}{2\omega_1(\omega_2+\varepsilon\Omega)^2}\left(\frac{KT}{m}\right)$$

$$\rho_4^0 = \frac{i}{(\omega_2+\varepsilon\Omega)}\sqrt{\frac{KT}{m}}\left[(1-\delta_1^+)\frac{K_x}{\omega_1} - \frac{K_2}{(\omega_2+\varepsilon\Omega)}\frac{E_{1z}}{E_{1x}}\right]$$

$$\rho_4^1 = i\sqrt{\frac{m}{KT}}\frac{1}{(\omega_2+\varepsilon\Omega)}\left[\frac{E_{1z}}{E_{1x}} + \frac{KT}{m}\frac{(1-\delta_1^+)K_x K_2}{\omega_1(\omega_2+\varepsilon\Omega)}\right]$$

$$\rho_4^2 = \frac{i K_2}{(\omega_2+\varepsilon\Omega)^2}\sqrt{\frac{m}{KT}}\frac{E_{1z}}{E_{1x}} \qquad \ldots(2.87)$$

$$\rho_5^0 = \frac{i}{(\omega_2+\varepsilon\Omega)}\sqrt{\frac{KT}{m}}(2-\delta_1^+)$$

$$\rho_5^1 = \frac{i}{(\omega_2+\varepsilon\Omega)}\sqrt{\frac{KT}{m}}\left[\frac{K_x}{\omega_1}\frac{E_{1z}}{E_{1x}} - (2-\delta_1^+)\left(\frac{K_{1z}}{\omega_1} - \frac{K_2}{(\omega_2+\varepsilon\Omega)}\right)\right]$$

$$\alpha_{11} = -(\rho_2^0 + \rho_2^1 V_z)\frac{m}{KT}\left[1 - \frac{\varepsilon\delta_1^+}{2}\left(e^{i(1-\varepsilon)\Omega\tau} - e^{-i(1+\varepsilon)\Omega\tau}\right)\right] - \frac{m}{KT}\rho_3\frac{E_{1z}}{E_{1x}}\left(e^{i(1-\varepsilon)\Omega\tau} + e^{-i(1+\varepsilon)\Omega\tau}\right)$$

$$\alpha_{12} = -(\rho_2^0 + \rho_2^1 V_z)\frac{m}{KT}\left[1 - \frac{\delta_1^+}{2}\left(e^{i(1-\varepsilon)\Omega\tau} + e^{-i(1+\varepsilon)\Omega\tau}\right)\right] - \frac{m}{KT}\varepsilon\rho_3\frac{E_{1z}}{E_{1x}}\left(e^{i(1-\varepsilon)\Omega\tau} - e^{-i(1+\varepsilon)\Omega\tau}\right)$$

$$\alpha_{13} = \delta_1^+\frac{m}{KT}(\rho_2^0 + \rho_2^1 V_z)e^{-i2\varepsilon\Omega\tau} + 2\frac{m}{KT}\rho_3\frac{E_{1z}}{E_{1x}}e^{-2i\varepsilon\Omega\tau}$$

$$\alpha_{14} = -(\rho_4^0 + \rho_4^1 V_z + \rho_4^2 V_z^2)\sqrt{\frac{m}{KT}}\,e^{-i\varepsilon\Omega\tau} = -\alpha_{14}$$

$$\alpha_{16} = \sqrt{\frac{m}{KT}}(\rho_5^0 + \rho_5^1 V_z)$$

If ϕ is defined as follows:

$$\phi = -\lambda(1-\cos\Omega\tau) + i(\omega - K_z v_z)\tau \qquad \ldots(2.88)$$

the results from equations (2.51) to (2.53) may be used to express the velocity moments of f_{31} over v_x and v_y as follows:

$$\langle f_{31}\rangle_\perp = \frac{Z^2e^2}{mKT} E_{1x}E_{2x}^* e^{i(\underline{K}\cdot\underline{r}-\omega t)} \int_0^\infty d\tau\, g(v_z)\, e^\phi \Big\{ \alpha_{11}\Big(\frac{KT}{m} - a^2\Big(\frac{KT}{m}\Big)^2\Big) + \alpha_{12}\Big(\frac{KT}{m} - \Big(\frac{bKT}{m}\Big)^2\Big) - i\alpha_{13}\, ab\Big(\frac{KT}{m}\Big)^2 + i\alpha_{14}\frac{aKT}{m} - \frac{bKT}{m}\alpha_{15} + \alpha_{16}\Big\} \qquad \ldots(2.89)$$

$$\langle v_x f_{31}\rangle_\perp = \frac{Z^2e^2}{mKT} E_{1x}E_{2x}^* e^{i(\underline{K}\cdot\underline{r}-\omega t)} \int_0^\infty d\tau\, g(v_z)\, e^\phi \Big\{ i\alpha_{11}\Big(b\Big(\frac{KT}{m}\Big)^2 - a^2b\Big(\frac{KT}{m}\Big)^3\Big) + i\alpha_{12}\Big(a\Big(\frac{KT}{m}\Big)^2 - a^2b\Big(\frac{KT}{m}\Big)^3\Big) - \alpha_{13}\Big(a\Big(\frac{KT}{m}\Big)^2 - ab^2\Big(\frac{KT}{m}\Big)^3\Big) + \alpha_{14}\Big(\frac{KT}{m} - \Big(\frac{aKT}{m}\Big)^2\Big) - i\alpha_{15}\, ab\Big(\frac{KT}{m}\Big)^2 + i\alpha_{16}\frac{aKT}{m}\Big\} \qquad \ldots(2.90)$$

$$\langle v_y f_{31}\rangle_\perp = \frac{Z^2e^2}{mKT} E_{1x}E_{2x}^* e^{i(\underline{K}\cdot\underline{r}-\omega t)} \int_0^\infty d\tau\, g(v_z)\, e^\phi \Big\{ i\alpha_{11}\Big(b\Big(\frac{KT}{m}\Big)^2 - a^2b\Big(\frac{KT}{m}\Big)^3\Big) + i\alpha_{12}\Big(3b\Big(\frac{KT}{m}\Big)^2 - \Big(\frac{bKT}{m}\Big)^2\Big) - \alpha_{13}\Big(a\Big(\frac{KT}{m}\Big)^2 - ab^2\Big(\frac{KT}{m}\Big)^3\Big) - \alpha_{14}\, ab\Big(\frac{KT}{m}\Big)^2 + i\alpha_{15}\Big(\frac{KT}{m} - \Big(\frac{bKT}{m}\Big)^2\Big) + i\alpha_{16}\, b\frac{KT}{m}\Big\} \qquad \ldots(2.91)$$

In order that terms with a common exponent can be collected so that the remaining integrations over v_z and τ can be performed, the various coefficients such as $a^2(kT/m)^2$ appearing in equations (2.89) to (2.91) must be evaluated. These are obtained by replacing $(\tau + \tau')$ by τ in

equation (2.57). By the use of the Bessel function equality given by equation (2.56), the velocity moments of f_{31} given by equations (2.89) to (2.91) may be expressed in an infinite series as follows:

$$\langle f_{31} \rangle_{\perp} = \frac{-Z^2e^2}{mKT} E_{1x} E_{2x}^{*} e^{i(\underline{K}\cdot\underline{r}-\omega t)} e^{-\lambda} \sum_{n=-\infty}^{\infty} \int_0^{\infty} d\tau\, g(v_z) \exp i\left[\omega - K_z v_z + n\Omega\right]\tau$$

$$\Big\{ (\rho_2^0 + \rho_2^1 v_z)\Big[2I_n - \frac{\delta_1^+}{2}\Big((1+\varepsilon) I_{n+(\varepsilon-1)} + (1-\varepsilon) I_{n+(1+\varepsilon)}\Big) + \frac{\lambda}{4}\Big(-8I_n$$

$$+ 4I_{n-1} + 4I_{n+1} - \frac{\delta_1^+}{2}\Big(8I_{n+\varepsilon} - (7+3\varepsilon) I_{n+\varepsilon-1} - (7-3\varepsilon) I_{n+(1+\varepsilon)} + 4I_{n+\varepsilon-2}$$

$$+ 4I_{n+(2+\varepsilon)} - (1-\varepsilon) I_{n+\varepsilon-3} - (1+\varepsilon) I_{n+(3+\varepsilon)} + 4\varepsilon I_{n+2\varepsilon-1} - 4\varepsilon I_{n+(1+2\varepsilon)}$$

$$- 2\varepsilon I_{n+2(\varepsilon-1)} + 2\varepsilon I_{n+2(1+\varepsilon)}\Big)\Big)\Big] + \rho_3 \frac{E_{1z}}{E_{1x}}\Big[(1+\varepsilon) I_{n+\varepsilon-1} + (1-\varepsilon) I_{n+(1+\varepsilon)}$$

$$+ \frac{\lambda}{4}\Big(-(1+5\varepsilon) I_{n+\varepsilon-1} - (1-5\varepsilon) I_{n+\varepsilon+1} + 4\varepsilon I_{n+\varepsilon-2} - 4\varepsilon I_{n+\varepsilon+2}$$

$$+ (1-\varepsilon) I_{n+\varepsilon-3} + (1+\varepsilon) I_{n+3+\varepsilon} - 4\varepsilon I_{n+2\varepsilon-1} + 4\varepsilon I_{n+1+2\varepsilon} + 2\varepsilon I_{n+2(\varepsilon-1)}$$

$$- 2\varepsilon I_{n+2(1+\varepsilon)}\Big)\Big] - \gamma(\rho_4^0 + \rho_4^1 v_z + \rho_4^2 v_z^2)\sqrt{\frac{m}{KT}}\Big[(1+\varepsilon) I_{n+\varepsilon-1} - (1-\varepsilon) I_{n+(1+\varepsilon)}$$

$$- 2\varepsilon I_{n-\varepsilon}\Big] - \sqrt{\frac{m}{KT}}\, (\rho_5^0 + \rho_5^1 v_z) I_n \Big\} \qquad \ldots(2.92)$$

$$\langle v_x f_{31} \rangle_{\perp} = \frac{Z^2e^2}{mKT} E_{1x} E_{2x}^{*} e^{i(\underline{K}\cdot\underline{r}-\omega t)} \gamma e^{-\lambda} \sum_{n=-\infty}^{\infty} \int_0^{\infty} d\tau\, g(v_z) \exp i\left[\omega - K_z v_z + n\Omega\right]$$

$$\Big\{ (\rho_2^0 + \rho_2^1 v_z)\Big[8\frac{m}{\lambda} I_n - \frac{\delta_1^+}{2}\Big(-6\varepsilon I_{n+\varepsilon} + (1+3\varepsilon) I_{n+\varepsilon-2} - (1-3\varepsilon) I_{n+(2+\varepsilon)}$$

$$+ 4\varepsilon I_{n+2\varepsilon} - 2\varepsilon I_{n+2\varepsilon-1} - 2\varepsilon I_{n+(1+2\varepsilon)}\Big) + \frac{\lambda}{4}\Big(-8I_{n-1} + 8I_{n+1} + 4I_{n-2}$$

$$- 4I_{n+2} - \frac{\delta_1^+}{2}\Big(6\varepsilon I_{n+\varepsilon} + 4I_{n+\varepsilon-1} - 4I_{n+(1+\varepsilon)} - (6+4\varepsilon) I_{n+\varepsilon-2}$$

$$+ (6-4\varepsilon) I_{n+(2+\varepsilon)} + 4I_{n+\varepsilon-3} - 4I_{n+(3+\varepsilon)} - (1-\varepsilon) I_{n+\varepsilon-4} + (1+\varepsilon) I_{n+\varepsilon+4}$$

$$-8\varepsilon I_{n+2\varepsilon} + 2\varepsilon I_{n+2\varepsilon-1} + 2\varepsilon I_{n+(1+2\varepsilon)} + 4\varepsilon I_{n+2(\varepsilon-1)} + 4\varepsilon I_{n+2(1+\varepsilon)}$$

$$-2\varepsilon I_{n+2\varepsilon-3} - 2\varepsilon I_{n+(3+2\varepsilon)}\Big)\Big)\Big] + \rho_3 \frac{E_{1z}}{E_{1x}}\Big[(3+\varepsilon) I_{n+\varepsilon-2} - (3-\varepsilon) I_{n+(2+\varepsilon)}$$

$$-2\varepsilon I_{n+\varepsilon} - 4\varepsilon I_{n+2\varepsilon} + 2\varepsilon I_{n+2\varepsilon-1} + 2\varepsilon I_{n+(1+2\varepsilon)} + \frac{\lambda}{4}\Big(10\varepsilon I_{n+\varepsilon} - 4\varepsilon I_{n+\varepsilon-1}$$

$$-4\varepsilon I_{n+(1+\varepsilon)} - 2(1+2\varepsilon) I_{n+\varepsilon-2} + 2(1-2\varepsilon) I_{n+(2+\varepsilon)} + 4\varepsilon I_{n+\varepsilon-3} + 4\varepsilon I_{n+(3+\varepsilon)}$$

$$+(1-\varepsilon) I_{n+\varepsilon-4} - (1+\varepsilon) I_{n+(4+\varepsilon)} + 8\varepsilon I_{n+2\varepsilon} - 2\varepsilon I_{n+2\varepsilon-1} - 2\varepsilon I_{n+(1+2\varepsilon)}$$

$$-4\varepsilon I_{n+2(\varepsilon-1)} - 4\varepsilon I_{n+2(1+\varepsilon)} + 2\varepsilon I_{n+2\varepsilon-3} + 2\varepsilon I_{n+(3+2\varepsilon)}\Big)\Big] - \sqrt{\frac{KT}{m}}\frac{1}{\gamma}\Big(\rho_4^0$$

$$+ \rho_4^1 v_z + \rho_4^2 v_z^2\Big)\Big[I_{n+\varepsilon} + \frac{\lambda}{4}\Big(-2I_{n+\varepsilon} - 2\varepsilon I_{n+\varepsilon-1} + 2\varepsilon I_{n+1+\varepsilon} + (1+\varepsilon) I_{n+\varepsilon-2}$$

$$+ (1-\varepsilon) I_{n+(2+\varepsilon)}\Big)\Big] - \sqrt{\frac{m}{KT}}\left(\rho_5^0 + \rho_5^1 v_z\right)\left(I_{n-1} - I_{n+1}\right)\Big\} \qquad \ldots(2.93)$$

$$\langle v_y f_{31}\rangle_\perp = \frac{-i\varepsilon Z^2 e^2}{mKT} E_{1x} E_{2x}^* e^{i(\underline{K}\cdot\underline{r}-\omega t)} \gamma e^{-\lambda} \sum_{n=-\infty}^{\infty} \int_0^\infty d\tau\, g(v_z) \exp i\left[\omega - K_z v_z + n\Omega\right]\tau$$

$$\Big\{\left(\rho_2^0 + \rho_2^1 v_z\right)\Big[8I_n - 4I_{n-1} - 4I_{n+1} - \frac{\delta_1^+}{2}\Big((16+2\varepsilon) I_{n+\varepsilon-1} + (6-2\varepsilon) I_{n+(1+\varepsilon)}$$

$$-6I_{n+\varepsilon} - (3+\varepsilon) I_{n+\varepsilon-2} - (3-\varepsilon) I_{n+(2+\varepsilon)} - 2\varepsilon I_{n+2\varepsilon-1} + 2\varepsilon I_{n+1+2\varepsilon}\Big) + \frac{\lambda}{4}\Big(24 I_n$$

$$+16 I_{n-1} + 16 I_{n+1} - 4I_{n-2} - 4I_{n+2} - \frac{\delta_1^+}{2}\Big(30 I_{n+\varepsilon} - (26+6\varepsilon) I_{n+\varepsilon-1}$$

$$-(26-6\varepsilon) I_{n+(1+\varepsilon)} + (16+2\varepsilon) I_{n+\varepsilon-2} + (16-2\varepsilon) I_{n+(2+\varepsilon)} - (6-2\varepsilon) I_{n+\varepsilon-3}$$

$$-(6+2\varepsilon) I_{n+\varepsilon-3} + (1-\varepsilon) I_{n+\varepsilon-4} + (1+\varepsilon) I_{n+(4+\varepsilon)} + 10\varepsilon I_{n+2\varepsilon-1} - 10\varepsilon I_{n+1+2\varepsilon}$$

$$-8\varepsilon I_{n+2(\varepsilon-1)} + 8\varepsilon I_{n+2(1+\varepsilon)} + 2\varepsilon I_{n+2\varepsilon-3} - 2\varepsilon I_{n+(3+2\varepsilon)}\Big)\Big)\Big]$$

$$+ \rho_3 \frac{E_{1z}}{E_{1x}}\Big[-2I_{n+\varepsilon} + (2+6\varepsilon) I_{n+\varepsilon-1} + (2-6\varepsilon) I_{n+(1+\varepsilon)} + (1+3\varepsilon) I_{n+\varepsilon-2}$$

$$-(1-3\varepsilon) I_{n+(2+\varepsilon)} + 2\varepsilon I_{n+2\varepsilon-1} - 2\varepsilon I_{n+(1+2\varepsilon)} + \frac{\lambda}{4}\Big(2I_{n+\varepsilon} - (2+14\varepsilon) I_{n+\varepsilon-1}$$

$$- (2-14\varepsilon) I_{n+(1+\varepsilon)} + 14\varepsilon I_{n+\varepsilon-2} - 14\varepsilon I_{n+2+\varepsilon} + (2-6\varepsilon) I_{n+\varepsilon-3} + (2+6\varepsilon) I_{n+3+\varepsilon}$$

$$-(1-\varepsilon) I_{n+\varepsilon-4} - (1+\varepsilon) I_{n+(4+\varepsilon)} - 10\varepsilon I_{n+2\varepsilon-1} + 10\varepsilon I_{n+(1+2\varepsilon)} + 8\varepsilon I_{n+2(\varepsilon-1)}$$

$$-8\varepsilon I_{n+2(1+\varepsilon)} - 2\varepsilon I_{n+2\varepsilon-3} + 2\varepsilon I_{n+(3+2\varepsilon)}\Big)\Big] - \frac{1}{\gamma}\sqrt{\frac{KT}{m}}\left(\rho_4^0 + \rho_4^1 v_z\right)\Big[\varepsilon I_{n+\varepsilon}$$

$$+ \frac{\lambda}{4}\Big(-6\varepsilon I_{n+\varepsilon} + (2+4\varepsilon) I_{n+\varepsilon-1} - (2-4\varepsilon) I_{n+(1+\varepsilon)} - (1+\varepsilon) I_{n+\varepsilon-2} + (1-\varepsilon) I_{n+(2+\varepsilon)}\Big)\Big]$$

$$- \sqrt{\frac{m}{KT}}\left(\rho_5^0 + \rho_5^1 v_z\right)\left(2I_n - I_{n-1} - I_{n+1}\right)\Big\} \qquad \ldots(2.94)$$

The integration with respect to τ and v_z, still to be performed in equations (2.92) to (2.94) are of the same type considered in equation (2.66) for the velocity moments of f_{32}. By using the results obtained for the integral in equation (2.66), and the notation for the driving currents as given by equation (2.79), the Π coefficients, evaluated to order zero in $\lambda^{\pm}$ for k = 1 may be expressed as follows:

$$\Pi^{+(1)}_{3(0)} = -\frac{(2-\delta_1^+)}{\gamma}\left(\rho_2^0 F_1 + \rho_2^1 F_2\right) - \frac{2F_1}{\gamma}\rho_3 \frac{E_{1z}}{E_{1x}} + 2\sqrt{\frac{m}{KT}}\left(\rho_4^0 F_1 + \rho_4^1 F_2 + \rho_4^2 F_3\right)$$

$$+ \frac{1}{\gamma}\sqrt{\frac{m}{KT}}\left(\rho_5^0 F_1 + \rho_5^1 F_2\right)$$

$$\Pi^{+(1)}_{3(-1)} = -2\sqrt{\frac{m}{KT}}\left(\rho_4^0 + \rho_4^1 F_2 + \rho_4^2 F_3\right) + \frac{2K_x}{\Omega_+}\left[\left(\rho_2^0 F_1 + \rho_2^1 F_2\right)\left(1 - \frac{3\delta_1^+}{2}\right) - \rho_3 \frac{E_{1z}}{E_{1x}} F_1\right]$$

$$\Pi^{-(1)}_{3(0)} = -(2-\delta_1^+)\left(\rho_2^0 F_1 + \rho_2^1 F_2\right) - 2\frac{\rho_3}{\gamma}\frac{E_{1z}}{E_{1x}} F_1 - 2\sqrt{\frac{m}{KT}}\left(\rho_4^0 F_1 + \rho_4^1 F_2 + \rho_4^2 F_3\right)$$

$$+ \frac{1}{\gamma}\sqrt{\frac{m}{KT}}\left(\rho_5^0 F_1 + \rho_5^1 F_2\right)$$

$$\Pi^{-(1)}_{3(1)} = 2\sqrt{\frac{m}{KT}}\left(\rho_4^0 F_1 + \rho_4^1 F_2 + \rho_4^2 F_3\right)$$

$$\Pi^{+(1)}_{1(1)} = 2(2-\delta_1^+)(\rho_2^0 F_0 + \rho_2^1 F_1) + 4\rho_3 \frac{E_{1z}}{E_{1x}} F_0 - \sqrt{\frac{m}{KT}}(\rho_5^0 F_0 + \rho_5^1 F_1)$$

$$\Pi^{+(1)}_{1(-1)} = -4(1-\delta_1^+)(\rho_2^0 F_0 + \rho_2^1 F_1) + \sqrt{\frac{m}{KT}}(\rho_5^0 F_0 + \rho_5^1 F_1) - \frac{1}{\gamma}\sqrt{\frac{KT}{m}}(\rho_4^0 F_0 + \rho_4^1 F_1 + \rho_4^2 F_2)$$

$$\Pi^{+(1)}_{1(-2)} = -2\delta_1^+(\rho_2^0 F_0 + \rho_2^1 F_1) - 4\rho_3 \frac{E_{1z}}{E_{1x}} F_0$$

$$\Pi^{-(1)}_{1(1)} = 4(1-\delta_1^+)(\rho_2^0 F_0 + \rho_2^1 F_1) - \frac{1}{\gamma}\sqrt{\frac{KT}{m}}(\rho_4^0 F_0 + \rho_4^1 F_1 + \rho_4^2 F_2) - \sqrt{\frac{m}{KT}}(\rho_5^0 F_0 + \rho_5^1 F_1)$$

$$\Pi^{-(1)}_{1(2)} = 2\delta_1^+(\rho_2^0 F_0 + \rho_2^1 F_1) + 4\rho_3 \frac{E_{1z}}{E_{1x}} F_0$$

$$\Pi^{-(1)}_{1(-1)} = -2(2-\delta_1^+)(\rho_2^0 F_0 + \rho_2^1 F_1) - 4\rho_3 \frac{E_{1z}}{E_{1x}} F_0 + \sqrt{\frac{m}{KT}}(\rho_5^0 F_0 + \rho_5^1 F_1)$$

$$\Pi^{+(1)}_{2(0)} = -i\left[4(2-\delta_1^+)(\rho_2^0 F_0 + \rho_2^1 F_1) + 8\rho_3 \frac{E_{1z}}{E_{1x}} F_0 - 2\sqrt{\frac{m}{KT}}(\rho_5^0 F_0 + \rho_5^1 F_1)\right]$$

$$\Pi^{+(1)}_{2(-1)} = -i\left[-4(1-\delta_1^+)(\rho_2^0 F_0 + \rho_2^1 F_1) - \sqrt{\frac{KT}{m}}\frac{1}{\gamma}(\rho_4^0 F_0 + \rho_4^1 F_1 + \rho_4^2 F_2) + \sqrt{\frac{m}{KT}}(\rho_5^0 F_0 + \rho_5^1 F_1)\right]$$

$$\Pi^{+(1)}_{2(-2)} = i\left[2\delta_1^+(\rho_2^0 F_0 + \rho_2^1 F_1) + 4\rho_3 \frac{E_{1z}}{E_{1x}} F_0\right]$$

$$\Pi^{-(1)}_{2(0)} = i\left[4(2-\delta_1^+)(\rho_2^0 F_0 + \rho_2^1 F_1) + 8\rho_3 \frac{E_{1z}}{E_{1x}} F_0 - 2\sqrt{\frac{m}{KT}}(\rho_5^0 F_0 + \rho_5^1 F_1)\right]$$

$$\Pi^{-(1)}_{-2(1)} = i\left[-4(1-\delta_1^+)(\rho_2^0 F_0 + \rho_2^1 F_1) + \frac{1}{\gamma}\sqrt{\frac{KT}{m}}(\rho_4^0 F_0 + \rho_4^1 F_1 + \rho_4^2 F_2) + \sqrt{\frac{m}{KT}}(\rho_5^0 F_0 + \rho_5^1 F_1)\right]$$

$$\Pi^{-(1)}_{2(2)} = -4i\rho_3 \frac{E_{1z}}{E_{1x}} F_0 - i2\delta_1^+(\rho_2^0 F_0 + \rho_2^1 F_1)$$

$$\Pi^{-(1)}_{2(-1)} = i\left[-2(2-\delta_1^+)\left(\rho_2^0 F_0 + \rho_2^1 F_1\right) - 4\rho_3 \frac{E_{1z}}{E_{1x}} F_0 + \sqrt{\frac{m}{KT}}\left(\rho_5^0 F_0 + \rho_5^1 F_1\right)\right] \qquad \ldots(2.95)$$

Equations (2.79), (2.80), and (2.95) may now be used to express the total second order driving current with the exponent $i(\underline{K}.\underline{r} - \omega t)$ as follows:

$$\underline{J}_{3d} = \sum_K \left(\underline{J}^{+K}_{3d} + \underline{J}^{-K}_{3d}\right) = \sum_K n_0 e\left[\langle \underline{v} f^+_{3K}\rangle - \langle \underline{v} f^-_{3K}\rangle\right] \qquad \ldots(2.96)$$

where k = 1, 2 (see equation (2.39)). If the ions and electrons are assumed to be at the same temperature, the total second order driving current may be expressed as follows:

$$J_{3d} = \frac{\omega D^-}{4\pi} \sum_{n=-\infty}^{\infty} \left(\sqrt{\frac{m_-}{m_+}}\, \Pi^+_{i(n)} - \Pi^-_{i(n)}\right) \hat{e}_i \qquad \ldots(2.97)$$

where $\Pi^{\pm}_{i(n)} = \Pi^{\pm(1)}_{i(n)} + \Pi^{\pm(2)}_{i(n)}$

$i = 1, 2, 3$ and denotes components in the x, y, and z directions respectively.

The $\Pi^{\pm}_{i(n)}$ coefficients are evaluated below through the use of equations (2.80) and (2.95).

$$\Pi^+_{3(0)} = iF_1\left[\frac{(2-\delta_1^+)}{\gamma}\frac{K_2 K_{1z}}{\omega_1\omega_2}\frac{KT}{m}\frac{\omega(\omega_1\omega_2 - \Omega_+^2)}{(\omega_2+\Omega_+)^2(\omega_1+\Omega_+)^2} - 2K_2\frac{E_{1z}}{E_{1x}}\frac{\omega(\omega_1\omega_2-\Omega_+^2)}{\omega_1\omega_2(\omega_1+\Omega_+)(\omega_2+\Omega_+)^2}\right.$$
$$\left. + \frac{(2-\delta_1^+)K_x}{\omega_1(\omega_2+\Omega_+)}\right] + iF_2\left[\frac{2m}{KT}\frac{E_{1z}}{E_{1x}}\frac{\omega}{\omega_1(\omega_2+\Omega_+)} + \frac{(2-\delta_1^+)}{\gamma}\left(\frac{K_2}{\omega_2(\omega_1+\Omega_+)}\right.\right.$$
$$\left.\left. - \frac{K_{1z}}{\omega_1(\omega_2+\Omega_+)}\right) + \frac{2(1-\delta_1^+)}{\omega_1}\frac{K_x K_2}{(\omega_2+\Omega_+)^2}\right] + \frac{iF_3}{\gamma}\left[\frac{K_x K_2}{\omega_1\omega_2(\omega_1+\Omega_+)}\right.$$
$$\left. + \frac{K_x}{\Omega_+}\left(\frac{K_2}{(\omega_2+\Omega_+)^2} - \frac{K_{1z}}{(\omega_1+\Omega_+)^2}\right)\right]\frac{E_{1z}}{E_{1x}}$$

$$\Pi^{+}_{3(-1)} = -iF_1 \Bigg\{ 2K_2 \frac{\left(-(\omega-\Omega_+)\omega_2 + \omega_2\Omega_+ + \Omega_+^2\right)}{\omega_1\omega_2\,(\omega_2+\Omega_+)^2}\frac{E_{1z}}{E_{1x}} + \frac{2(1-\delta_1^+)K_x}{\omega_1\,(\omega_2+\Omega_+)}$$

$$+ \frac{2K_x}{\Omega_+}\Bigg[(1-3/2\,\delta_1^+)\left(\frac{(\omega-\Omega_+)}{\omega_1(\omega_2+\Omega_+)} - \frac{K_2K_{1z}}{\omega_1}\frac{KT}{m}\frac{\left(\omega(\omega_1+\omega_2)-2\omega_2\Omega_+-\Omega_+^2\right)}{\omega_1^2\,(\omega_2+\Omega_+)^2}\right)$$

$$+ \frac{\delta_1^+}{2}\frac{\Omega_+}{\omega_1\,(\omega_1-\Omega_+)} - \frac{K_xK_2}{2\omega_1\,(\omega_2+\Omega_+)^2}\frac{KT}{m}\frac{E_{1z}}{E_{1x}}\Bigg]\Bigg\} - iF_2\Bigg\{ \frac{2m}{KT}\frac{(\omega-\Omega_+)}{\omega_1(\omega_2+\Omega_+)}\frac{E_{1z}}{E_{1x}}$$

$$+ \frac{2(1-\delta_1^+)K_xK_2}{\omega_1\,(\omega_2+\Omega_+)^2} + \frac{2K_x}{\Omega_+}\Bigg[\frac{K_2}{(\omega_2+\Omega_+)^2} - \frac{K_{1z}}{\omega_1^2} + \frac{\delta_1^+}{2}\Bigg(K_{1z}\left(\frac{1}{(\omega_1-\Omega_+)^2}\right.$$

$$\left.\left. - \frac{1}{\omega_1^2}\right) - \frac{K_2\Omega_+}{\omega_1\omega_2(\omega_1+\Omega_+)}\Bigg]\Bigg\} + i\frac{F_3}{\gamma}\frac{K_x}{\Omega_+}\left(\frac{K_{1z}}{\omega_1^2} - \frac{K_2}{(\omega_2+\Omega_+)^2}\right)\frac{E_{1z}}{E_{1x}}$$

$$\Pi^{-}_{3(0)} = 2iF_1\Bigg[(2-\delta_1^+)\frac{K_2K_{1z}\omega\Omega_-(\omega_1\omega_2-\Omega_-^2)}{\omega_1\omega_2\,(\omega_2-\Omega_-)^2\,(\omega_1-\Omega_-)} + K_2\frac{E_{1z}}{E_{1x}}\frac{\omega(\omega_1\omega_2-\Omega_+^2)}{\omega_1\omega_2\,(\omega_1-\Omega_-)(\omega_2-\Omega_-)^2}$$

$$- \frac{K_x(1-\delta_1^+)}{\omega_1\,(\omega_2-\Omega_-)}\Bigg] + 2iF_2\Bigg[-\frac{m}{KT}\frac{E_{1z}}{E_{1x}}\frac{\omega}{\omega_1(\omega_2-\Omega_-)} + \frac{(1-\delta_1^+/2)}{\gamma}\Bigg(\frac{K_2}{\omega_2(\omega_1-\Omega_-)}$$

$$- \frac{K_{1z}}{\omega_1(\omega_2-\Omega_-)}\Bigg) - \frac{(1-\delta_1^+)K_xK_2}{\omega_1\,(\omega_2-\Omega_-)^2}\Bigg] + i\frac{F_3}{\gamma}\frac{E_{1z}}{E_{1x}}\frac{K_x}{\Omega_-}\Bigg[\frac{\Omega_-K_2}{\omega_1\omega_2(\omega_1-\Omega_-)} + \frac{K_{1z}}{(\omega_1-\Omega_-)^2} - \frac{K_2}{(\omega_2-\Omega_-)^2}\Bigg]$$

$$\Pi^{-}_{3(1)} = 2iF_1\Bigg[\frac{K_2}{\omega_1\omega_2}\frac{E_{1z}}{E_{1x}}\left(1 - \frac{\omega_1\omega_2}{(\omega_2-\Omega_-)^2}\right) + \frac{(1-\delta_1^+)K_x}{\omega_1\,(\omega_2-\Omega_-)}\Bigg] + 2iF_2\Bigg[\frac{m}{KT}\frac{(\omega+\Omega_-)}{\omega_1(\omega_2-\Omega_-)}\frac{E_{1z}}{E_{1x}}$$

$$+ \frac{(1-\delta_1^+)K_xK_2}{\omega_1\,(\omega_2-\Omega_-)^2}\Bigg] - i\frac{F_3}{\gamma}\frac{K_x}{\Omega_-}\frac{E_{1z}}{E_{1x}}\left(\frac{K_{1z}}{\omega_1^2} - \frac{K_2}{(\omega_2-\Omega_-)^2}\right)$$

$$\Pi^{+}_{1(1)} = iF_0\Bigg[\frac{(2-\delta_1^+)\omega}{(\omega_1+\Omega_+)(\omega_2+\Omega_+)} + 2(2-\delta_1^+)K_2K_{1z}\frac{KT}{m}\left(\frac{1}{\omega_2(\omega_1+2\Omega_+)^2} - \frac{1}{\omega_1(\omega_2+\Omega_+)^2}\right)$$

$$+ \frac{2K_xK_2}{\omega_1(\omega_2+\Omega_+)^2}\frac{KT}{m}\frac{E_{1z}}{E_{1x}}\Bigg] + iF_1\Bigg[-K_x\frac{E_{1z}}{E_{1x}}\frac{\left(\omega\omega_1 + 2(\omega_1+\Omega_+)\right)}{\omega_1(\omega_1+\Omega_+)(\omega_1+2\Omega_+)(\omega_2+\Omega_+)}$$

$$+ (2-\delta_1^+)\left(\frac{K_{1z}}{\omega_1(\omega_2+\Omega_+)} - \frac{K_2\,\omega_1}{\omega_2(\omega_1+\Omega_+)(\omega_1+2\Omega_+)}\right) - (2-\delta_1^+)\Bigg(\frac{K_{1z}}{(\omega_1+\Omega_+)^2}$$

$$- \frac{K_2}{(\omega_2+\Omega_+)^2}\Bigg)\Bigg] - iF_2 \frac{E_{1z}}{E_{1x}}\left[\frac{K_x K_2}{\omega_2(\omega_1+\Omega_+)(\omega_1+2\Omega_+)} - \frac{K_x K_{1z}}{\Omega_+(\omega_1+\Omega_+)^2}\left(1 - \frac{(\omega_1+\Omega_+)^2}{(\omega_1+2\Omega_+)^2}\right)\right]$$

$$\Pi^+_{2(1)} = i\,\Pi^+_{1(1)}$$

$$\Pi^+_{1(-1)} = iF_0\Bigg[\frac{(-2\omega+\delta_1^+(3\omega-\Omega_+))}{\omega_1(\omega_2+\Omega_+)} + 2K_2\left(2(1-\delta_1^+)\frac{K_{1z}}{\omega_1}\frac{KT}{m} + \frac{\Omega_+ E_{1z}}{K_x E_{1x}}\right)\cdot$$

$$\left(\frac{\omega\omega_2 - 2\omega_2\Omega_+ - \Omega_+^2}{\omega_1\omega_2\,(\omega_2+\Omega_+)^2}\right) - \frac{\Omega_+\delta_1^+}{\omega_1(\omega_1-\Omega_+)}\Bigg] + iF_1\Bigg[\frac{K_x}{\omega_1}\frac{E_{1z}}{E_{1x}}\frac{(\omega-2\Omega_+)}{(\omega_1-\Omega_+)(\omega_2+\Omega_+)}$$

$$-\frac{2\Omega_+}{K_x}\frac{m}{KT}\frac{(\omega-\Omega_+)}{\omega_1(\omega_2+\Omega_+)}\frac{E_{1z}}{E_{1x}} - \frac{2\Omega_+ K_2 K_{1z}}{\omega_1^2\omega_2 K_x}\frac{E_{1z}}{E_{1x}} + \frac{(2-\delta_1^+)}{\omega_1}\left(\frac{K_2}{\omega_2} - \frac{K_{1z}}{(\omega_2+\Omega_+)}\right)$$

$$+\frac{\Omega_+\delta_1^+ K_2}{\omega_1\omega_2(\omega_1-\Omega_+)} + 2(1-\delta_1^+)\left(\frac{K_{1z}}{\omega_1^2} - \frac{K_2}{(\omega_2+\Omega_+)^2}\right) - \delta_1^+\left(\frac{K_{1z}}{(\omega_1-\Omega_+)^2} - \frac{K_2}{(\omega_2+\Omega_+)^2}\right)\Bigg]$$

$$+iF_2\left[\frac{K_x K_2}{\omega_1\omega_2(\omega_1-\Omega_+)} + \frac{1}{\gamma}\left(\frac{K_{1x}}{\omega_1^2} - \frac{K_2}{(\omega_2+\Omega_+)^2}\right) - \frac{K_x K_{1z}(2\omega_1-\Omega_+)}{\omega_1^2(\omega_1-\Omega_+)^2}\right]\frac{E_{1z}}{E_{1x}}$$

$$\Pi^+_{2(-1)} = \Pi^+_{1(-1)}$$

$$\Pi^+_{1(-2)} = -2iF_0\Bigg[\frac{(\omega-2\Omega_+)\delta_1^+}{(\omega_1-\Omega_+)(\omega_2+\Omega_+)} + \delta_1^+\frac{K_2K_{1z}}{\omega_1\omega_2}\frac{KT}{m}\left(\frac{\omega(\omega_1\omega_2-\Omega_+^2)}{(\omega_2+\Omega_+)^2(\omega_1-\Omega_+)^2}\right)$$

$$-\frac{K_xK_2}{\omega_1(\omega_2+\Omega_+)^2}\frac{KT}{m}\frac{E_{1z}}{E_{1x}}\Bigg] + 2i\,\delta_1^+F_1\left[\frac{K_{1z}}{(\omega_1-\Omega_+)^2} - \frac{K_2}{(\omega_2+\Omega_+)^2}\right]$$

$$\Pi^+_{2(-2)} = i\,\Pi^+_{1(-2)}$$

$$\Pi^+_{2(0)} = F_0\Bigg[\frac{2(2-\delta_1^+)\omega}{(\omega_1+\Omega_+)(\omega_2+\Omega_+)} - 4\frac{(2-\delta_1^+)K_2K_{1z}}{\omega_1\omega_2}\frac{KT}{m}\frac{\omega(\omega_1\omega_2-\Omega_+^2)}{(\omega_2+\Omega_+)^2(\omega_1+\Omega_+)^2}$$

$$+\frac{4K_xK_2}{\omega_1(\omega_2+\Omega_+)^2}\frac{KT}{m}\frac{E_{1z}}{E_{1x}}\Bigg] - 2F_1\Bigg[\frac{K_x}{\omega_1}\frac{E_{1z}}{E_{1x}}\frac{\omega}{(\omega_1+\Omega_+)(\omega_2+\Omega_+)} + (2-\delta_1^+)\Bigg(\frac{K_2}{\omega_2(\omega_1+\Omega_+)}$$

$$\left.\frac{-K_{1z}}{\omega_1(\omega_2+\Omega_+)}\right) + (2-\delta_1^+)\left(\frac{K_{1z}}{(\omega_1+\Omega_+)^2} - \frac{K_2}{(\omega_2+\Omega_+)^2}\right)\Bigg] - 2F_2\left[\frac{K_x K_2}{\omega_1\omega_2(\omega_1+\Omega_+)}\right.$$

$$\left. - \frac{K_x K_{1z}}{\omega_1^2\,\Omega_+}\left(1 - \frac{\omega_1^2}{(\omega_1+\Omega_+)^2}\right)\right]\frac{E_{1z}}{E_{1x}}$$

$$\Pi^-_{2(0)} = -F_0\left[\frac{2(2-\delta_1^+)\,\omega}{(\omega_2-\Omega_-)(\omega_1-\Omega_-)} + 4(2-\delta_1^+)K_2K_{1z}\frac{KT}{m}\frac{\omega(\Omega_-^2-\omega_1\omega_2)}{\omega_1\omega_2(\omega_1-\Omega_-)^2(\omega_2-\Omega_-)^2}\right.$$

$$\left. + \frac{4K_xK_2}{\omega_1(\omega_2-\Omega_-)^2}\frac{KT}{m}\frac{E_{1z}}{E_{1x}}\right] - 2F_1\left[(2-\delta_1^+)\left(\frac{K_{1z}}{\omega_1(\omega_2-\Omega_-)} - \frac{K_2}{\omega_2(\omega_1-\Omega_-)}\right)\right.$$

$$\left. - \frac{K_x}{\omega_1}\frac{E_{1z}}{E_{1x}}\frac{\omega}{(\omega_1-\Omega_-)(\omega_2-\Omega_-)} - (2-\delta_1^+)\left(\frac{K_{1z}}{(\omega_1-\Omega_-)^2} - \frac{K_2}{(\omega_2-\Omega_-)^2}\right)\right]$$

$$+ 2F_2\left[\frac{K_xK_2}{\omega_1\omega_2(\omega_1-\Omega_-)} + \frac{K_xK_{1z}}{\omega_1^2\Omega_-}\left(1 - \frac{\omega_1^2}{(\omega_1-\Omega_-)^2}\right)\right]\frac{E_{1z}}{E_{1x}}$$

$$\Pi^-_{1(1)} = iF_0\left[\frac{(2\omega - \delta_1^+(3\omega+\Omega_-))}{\omega_1(\omega_2-\Omega_-)} + \frac{2K_2}{\omega_1\omega_2}\left(1 - \frac{\omega_1\omega_2}{(\omega_2-\Omega_-)^2}\right)\left(2(1-\delta_1^+)\frac{K_{1z}}{\omega_1}\frac{KT}{m}\right.\right.$$

$$\left.\left. - \frac{\Omega_-}{K_x}\frac{E_{1z}}{E_{1x}}\right) - \frac{\Omega_-\delta_1^+}{\omega_1(\omega_1+\Omega_-)}\right] - iF_1\left[\frac{K_x}{\omega_1}\frac{E_{1z}}{E_{1x}}\left(\frac{1}{\omega_2-\Omega_-} - \frac{\omega_1}{(\omega_1-\Omega_-)(\omega_1-2\Omega_-)}\right)\right.$$

$$+ (2-\delta_1^+)\left(\frac{K_2\omega_1}{\omega_2(\omega_1-\Omega_-)(\omega_1-2\Omega_-)} - \frac{K_{1z}}{\omega_1(\omega_2-\Omega_-)}\right) - 2(1-\delta_1^+)\left(\frac{K_{1z}}{\omega_1^2}\right.$$

$$\left.\left. - \frac{K_2}{(\omega_2-\Omega_-)^2}\right) + \delta_1^+\left(\frac{K_{1z}}{(\omega_1+\Omega_-)^2} - \frac{K_2}{(\omega_2-\Omega_-)^2}\right)\right] - iF_2\left[\frac{K_xK_2}{\omega_1\omega_2(\omega_1+\Omega_-)}\right.$$

$$\left. - \frac{1}{\gamma}\left(\frac{K_{1z}}{\omega_1^2} - \frac{K_2}{(\omega_2-\Omega_-)^2}\right) - \frac{K_xK_{1z}}{\omega_1^2\Omega_-}\left(1 - \frac{\omega_1^2}{(\omega_1+\Omega_-)^2}\right)\right]\frac{E_{1z}}{E_{1x}}$$

$$\Pi^-_{2(1)} = -i\,\Pi^-_{1(1)}$$

$$\Pi^-_{1(-1)} = iF_0\left[\frac{-(2-\delta_1^+)\,\omega}{(\omega_1-\Omega_-)(\omega_2-\Omega_-)} + \frac{2(2-\delta_1^+)K_2K_{1z}}{\omega_1}\frac{KT}{m}\left(\frac{1}{(\omega_2-\Omega_-)^2} - \frac{\omega_1}{\omega_2(\omega_1-2\Omega_-)^2}\right)\right.$$

$$-\frac{2K_xK_2}{\omega_1(\omega_2-\Omega_-)^2}\frac{KT}{m}\frac{E_{1z}}{E_{1x}}\Bigg] + iF_1\Bigg[\frac{K_x}{\omega_1}\frac{E_{1z}}{E_{1x}}\left(\frac{1}{\omega_2-\Omega_-} - \frac{\omega_1}{(\omega_1-\Omega_-)(\omega_1-2\Omega_-)}\right)$$

$$+ (2-\delta_1^+)\left(\frac{K_2\,\omega_1}{\omega_2(\omega_1-\Omega_-)(\omega_1-2\Omega_-)} - \frac{K_{1z}}{\omega_1(\omega_2-\Omega_-)}\right) + (2-\delta_1^+)\left(\frac{K_{1z}}{(\omega_1-\Omega_-)^2} - \frac{K_2}{(\omega_2-\Omega_-)^2}\right)\Bigg]$$

$$+ iF_2\left[\frac{K_xK_2}{\omega_2(\omega_1-\Omega_-)(\omega_1-2\Omega_-)} + \frac{K_{1z}K_x}{\Omega_-(\omega_1-\Omega_-)^2}\left(1-\frac{(\omega_1-\Omega_-)^2}{(\omega_1-2\Omega_-)^2}\right)\right]\frac{E_{1z}}{E_{1x}}$$

$$\Pi^-_{2(-1)} = i\,\Pi^-_{1(-1)}$$

$$\Pi^-_{1(2)} = 2iF_0\left[\frac{\delta_1^+(\omega+2\Omega_-)}{(\omega_2-\Omega_-)(\omega_1+\Omega_-)} - \delta_1^+\frac{K_2K_{1z}}{\omega_1}\frac{KT}{m}\left(\frac{1}{(\omega_2-\Omega_-)^2} - \frac{\omega_1}{\omega_2(\omega_1+\Omega_-)^2}\right)\right.$$

$$\left.+\frac{K_xK_2}{\omega_1(\omega_2-\Omega_-)^2}\frac{KT}{m}\frac{E_{1z}}{E_{1x}}\right] - 2i\delta_1^+F_1\left[\frac{K_{1z}}{(\omega_1+\Omega_-)^2} - \frac{K_2}{(\omega_2-\Omega_-)^2}\right]$$

$$\Pi^-_{2(2)} = -i\,\Pi^-_{1(2)}$$

...(2.98)

2.3.3 Velocity Moments of f_{33}

The results for the induced second order current, namely the current that is obtained from the velocity moments of f_{33} in equation (2.39), may be obtained directly from the results given by Stix[10] (section 9.2). In the analysis by Stix[10], the currents are obtained by first defining a mobility tensor as follows:

$$\langle \underline{v} \rangle^{\pm} = \frac{c}{B_o} \underline{\underline{M}}_3^{\pm} \cdot \underline{E}_3 \qquad \text{...(2.99)}$$

As in Stix[10] (section 10.8), a correction to the mobility tensor will be made to account for collisional effects between the species under consideration. Before this can be accomplished, the mobility tensor for a collisionless plasma must be obtained. Terms of order λ will be neglected in the evaluation of the mobility tensor below. The terms F_o, F_1, ... (where F_p is given by equation (2.66)) will be denoted as $F_{o(n)}$, $F_{1(n)}$, ... where n refers to the α_n used in evaluating these terms. The various terms in the mobility tensor are written so that they will be applicable to ions or electrons by the appropriate substitutions for ε, m, T, and Ω.

$$M_{3xx} = \frac{\varepsilon\Omega}{2K_z}\sqrt{\frac{m}{2KT}}\left(F_{o(1)} + F_{o(-1)}\right)$$

$$M_{3xy} = \frac{i\Omega}{2K_z}\sqrt{\frac{m}{2KT}}\left(F_{o(1)} - F_{o(-1)}\right)$$

$$M_{3xz} = -\frac{\varepsilon K_x}{K_z}\sqrt{\frac{m}{2KT}}\left(F_{1(1)} - F_{1(-1)}\right) = M_{3xz}$$

$$M_{3yx} = -M_{3xy} \qquad \text{...(2.100)}$$

$$M_{3yy} = M_{3xx}$$

$$M_{3yz} = \frac{iK_x}{2K_z}\sqrt{\frac{m}{2KT}}\left[2F_{1(0)} - F_{1(1)} - F_{1(-1)}\right]$$

$$M_{3zx} = M_{3xz}$$

$$M_{3zz} = \frac{\varepsilon\Omega}{K_z}\left(\frac{m}{2KT}\right)^{3/2}\left[2F_{2(0)} + \lambda F_{2(-1)}\right]$$

where M_{3ij} is the tensor component relating the velocity component in the i'th direction to the electric field component in the j'th direction. By the use of equations (2.97) and (2.99), the total second order current may be written as follows:

$$J_3 = \frac{\omega_{p-}^2}{4\pi\Omega} \underline{\underline{M}}_3 \cdot \underline{E}_3 + \frac{\omega}{4\pi} \underline{J}'_{3d} \qquad \ldots(2.101)$$

where $\underline{\underline{M}}_3 = \underline{\underline{M}}_3^+ - \underline{\underline{M}}_3^-$

$$\underline{J}'_{3d} = \frac{4\pi}{\omega} \underline{J}_{3d}$$

2.4 Correction to Mobility Tensor for Ion-electron Collisional Effects

The mobility tensor will now be corrected for ion-electron collisional effects. When a resonance in the second order current is obtained by allowing the difference frequency ω to approach the ion cyclotron frequency, collisional effects will be relatively unimportant[26] if $\nu_{ie} << \sqrt{\frac{kT}{m}} K_z$ (where $\nu_{ie} = m_-/m_+ \nu_{ei}$). This may be seen by considering the mobility tensor component M^+_{3xx}. For a collisionless plasma,

$$M^+_{3xx}\,(\text{collisionless}) = \frac{i\Omega_+}{2}\left[\frac{1}{\omega+\Omega_+} + \frac{1}{\omega-\Omega_+} - i\frac{\sqrt{\pi}}{K_z}\sqrt{\frac{m_+}{2KT^+}}\,\mathrm{sgn}(K_z)\exp(\alpha_{-1}^2)\right] \text{ for } |\alpha_{-1}| > 1$$

$$= \frac{i\Omega_+}{2}\left[\frac{1}{\omega+\Omega_+} - i\frac{\sqrt{\pi}}{K_z}\sqrt{\frac{m_+}{2KT^+}} - \frac{2(\omega-\Omega_+)}{K_z^2}\frac{m_+}{2KT^+}\right] \text{ for } |\alpha_{-1}| < 1$$

$$\ldots(2.102)$$

The expression that is obtained for M^{+}_{3xx} from a cold plasma analysis, with collisional effects included, is

$$M^{+}_{3xx}(\text{cold}) = \frac{i\Omega_+}{2}\left[\frac{(\omega+\Omega_-)}{(\omega+\Omega_-)(\omega-\Omega_+)+i\omega\nu} + \frac{(\omega-\Omega_-)}{(\omega-\Omega_-)(\omega+\Omega_+)+i\omega\nu}\right] \qquad \ldots(2.103)$$

where $\nu = \nu_{ei}(1 + m_-/m_+)$. As ω approaches the ion cyclotron frequency, a resonance in both the collisionless and cold plasma expressions for M^{+}_{3xx} is obtained. However, the magnitude of this resonance will be bounded by cyclotron damping effects for $(\omega - \Omega_+) \lesssim (2KT/m_+)^{1/2}K_z$ in the case of a collisionless plasma (equation (2.102), or by collisional effects for $(\omega - \Omega_+) \lesssim \nu_{ie}$ (equation (2.103)) in the case of a plasma where collisional effects are dominant. Therefore, if $\nu_{ie} << (2KT^{+}/m_+)^{1/2}$, cyclotron damping will be the dominant mechanism for ion energy absorption from the difference frequency wave.

For the case that $\nu_{ie} >> (2KT^{+}/m_+)^{1/2}K_z$, the absorption of energy by the ions from the second order fields will be determined exclusively by collisions. This will occur for very dense, low temperature plasmas. However, for this case there is no need for a kinetic analysis, since a cold plasma analysis will suffice[26]. This case will not be considered in this thesis, since the use of cyclotron damping presents a much more powerful technique for the heating of high temperature plasmas.

Outside the "inner resonance region", that is for $(\omega - \Omega_+) > (2KT^{+}/m_+)^{1/2}K_z$, collisional effects will be included in the mobility tensor by the use of the cold plasma theory as described by Stix[10] (section (10.8)). That is, for the case that $\nu_{ie} << (2KT^{+}/m_+)^{1/2}K_z$, collisional effects will only serve to broaden the frequency range for

resonant ion energy absorption.

In the cold plasma analysis of the nonlinear mixing of HF waves by James and Thompson[17], collisional effects introduce terms of the order $\leq \nu_{ie}/(\omega - \Omega_+)$ into the expressions for the second order driving currents. The net effect of these terms in the expression for the ion power absorption from the second order fields is of the order $(\nu_{ie}/(\omega - \Omega_+))^2$. For the case of whistler heating, a cold plasma analysis will give similar results, namely that collisional effects introduce terms of the order $\lesssim (\nu_{ie}/(\omega - \Omega_+))$ into the second order driving currents. To check if similar results are applicable to whistler heating, the cold plasma equations were programmed on the IBM 360 computer. An error of one percent in the ion power absorption was introduced when collisional effects were neglected in the driving terms for a plasma with a density of $10^{16}/cm^3$, a temperature of 10^6 ^oK, and with $(\omega - \Omega_+) \simeq 10\ \nu_{ie}$. This is what the results for the nonlinear mixing of HF waves predict.

The method used by Stix[10] for including collisional effects in the mobility tensor is outlined below. The analysis begins with the conservation of momentum equation for species j.

$$n_j m_j \frac{d\underline{v}_j}{dt} = n_j e \varepsilon_j \left(\underline{E} + \frac{\underline{v}_j \times \underline{B}}{c}\right) + n_j m_j \sum_K \underline{\underline{\delta}}_{jK} \cdot (\underline{v}_K - \underline{v}_j) \qquad \ldots(2.104)$$

where the last term on the RHS of the above equation represents the effect of collisions with species k. The effect of collisions will be accounted for by introducing an additional velocity component $\Delta\underline{v}_j$. By the use of the mobility tensor notation given by equation (2.99), the

introduction of $\Delta \underline{v}_j$ into equation (2.104) yields

$$\left(\underline{v}_j + \Delta \underline{v}_j \right) = \left(\underline{\underline{M}}_3^j + \Delta \underline{\underline{M}}_3^j \right) \cdot \underline{E}_3$$

from which $\Delta \underline{\underline{M}}_3^j$ may easily be shown to be

$$\Delta \underline{\underline{M}}_3^j = \frac{1}{\varepsilon_j \Omega_j} \sum_K \underline{\underline{M}}_3^j \cdot \underline{\underline{\nu}}_{jK} \cdot \left(\underline{\underline{M}}_3^K - \underline{\underline{M}}_3^j \right)$$

where $\underline{\underline{\nu}}_{jk} = \nu_{jk} \underline{\underline{I}}$. If the conservation of momentum is to hold for the various species, the following relation must hold.

$$n_j m_j \nu_{jK} = n_K m_K \nu_{Kj} \qquad \text{...(2.105)}$$

For the case under consideration, namely for a plasma composed of two species (ions and electrons)

$$\Delta \underline{\underline{M}}_3^+ = \frac{\nu_{ie}}{\Omega_+} \underline{\underline{M}}_3^+ \cdot \underline{\underline{I}} \cdot \left(\underline{\underline{M}}_3^- - \underline{\underline{M}}_3^+ \right)$$

$$\Delta \underline{\underline{M}}_3^- = -\frac{\nu_{ei}}{\Omega_+} \underline{\underline{M}}_3^- \cdot \underline{\underline{I}} \cdot \left(\underline{\underline{M}}_3^+ - \underline{\underline{M}}_3^- \right) \qquad \text{...(2.106)}$$

where the mobility tensors on the RHS of equation (2.106) are those obtained by neglecting collisions. The technique used above may be shown to be equivalent to a cold plasma analysis in which just the first two terms in the asymptotic expansions of $\dfrac{1}{\left(1 + \dfrac{\omega \nu_{ei}(1 + m_-/m_+)}{(\omega - \Omega_+)(\omega + \Omega_-)}\right)}$

and $\dfrac{1}{\left(1 + \dfrac{\omega \nu_{ei}(1 + m_-/m_+)}{(\omega + \Omega_+)(\omega - \Omega_-)}\right)}$ are included.

2.5 Solution for the Second Order Field and Ion Energy Absorption

Equation (2.15) for the second order fields is

$$\left[\left(\frac{\omega}{c}\right)^2 - K^2\right] = -i\frac{4\pi}{\omega}\left[\left(\frac{\omega}{c}\right)^2 \underline{J}_3 - (\underline{K}\cdot\underline{J}_3)\,\underline{K}\right] \qquad \ldots(2.107)$$

Through the use of equations (2.99), (2.100), (2.106), and (2.107), the relationship between the second order electric field and the relevant currents may be expressed in matrix form as follows:

$$\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{bmatrix} E_{3x} \\ E_{3y} \\ E_{3z} \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \qquad \ldots(2.108)$$

where

$$Y_{11} = (1-n^2) + \frac{i\omega_{p-}^2}{\omega\Omega_-}\left((1-n_x^2)M_{3xx} - n_x n_z M_{3zx}\right)$$

$$Y_{12} = \frac{i\omega_{p-}^2}{\omega\Omega_-}\left((1-n_x^2)M_{3xy} - n_x n_z M_{3zy}\right)$$

$$Y_{13} = \frac{i\omega_{p-}^2}{\omega\Omega_-}\left((1-n_x^2)M_{3xz} - n_x n_z M_{3zz}\right)$$

$$R_1 = -i\left((1-n_x^2)J_{3dx} - n_x n_z J'_{3dz}\right) \qquad \ldots(2.109)$$

$$Y_{21} = \frac{i\omega_{p-}^2}{\omega\Omega_-}M_{3yx}$$

$$Y_{22} = (1-n^2) + \frac{i\omega_{p-}^2}{\omega\Omega_-}M_{3yy}$$

$$Y_{23} = \frac{i\omega_{p-}^2}{\omega\Omega_-}M_{3yz}$$

$$R_2 = -i J'_{3dy}$$

$$Y_{31} = \frac{i\omega_{p-}^2}{\omega\Omega_-}\left((1-n_z^2) M_{3zx} - n_x n_z M_{3xx}\right)$$

$$Y_{32} = \frac{i\omega_{p-}^2}{\omega\Omega_-}\left((1-n_z^2) M_{3zy} - n_x n_z M_{3xy}\right)$$

$$Y_{33} = (1-n^2) + \frac{i\omega_{p-}^2}{\omega\Omega_-}\left((1-n_z^2) M_{3zz} - n_x n_z M_{3xz}\right)$$

$$R_3 = -i(1-n_z^2) J'_{3dz} + i n_x n_z J'_{3dx}$$

$$n = c/\omega \; K$$

= refractive index of plasma for the difference frequency wave.

The various field components are then given by

$$E_{3j} = \sum_{K=1}^{3} R_K C_{Kj} (-1)^{j+K} / \det[Y] \qquad \text{...(2.110)}$$

where j = 1, 2, 3 and denotes components in the x, y, and z directions respectively. The C_{kj} terms are the cofactors for the matrix [Y] given by equation (2.108).

$$\det[Y] = \sum_{K=1}^{3} (-1)^{j+1} Y_{K1} C_{K1}$$

The various cofactors are:

$$C_{11} = Y_{22}Y_{33} - Y_{23}Y_{32} \qquad C_{12} = Y_{21}Y_{33} - Y_{23}Y_{31} \qquad C_{13} = Y_{21}Y_{32} - Y_{22}Y_{31}$$

$$C_{21} = Y_{12}Y_{33} - Y_{13}Y_{32} \qquad C_{22} = Y_{11}Y_{33} - Y_{13}Y_{31} \qquad C_{23} = Y_{11}Y_{32} - Y_{12}Y_{31}$$

$$C_{31} = Y_{12}Y_{23} - Y_{13}Y_{22} \qquad C_{32} = Y_{11}Y_{23} - Y_{13}Y_{21} \qquad C_{33} = Y_{11}Y_{22} - Y_{12}Y_{21}$$

...(2.111)

From equation (2.99), the induced second order ion current is

$$\underline{J}_3^+ = \frac{\omega_{p+}^2}{4\pi\Omega_+} \underline{\underline{M}}_3^+ \cdot \underline{E}_3 \qquad \ldots(2.112)$$

The energy absorbed by the ions from the second order electric field, $\underline{E}_3$, is given by

$$W_3^i = \text{Real}\ (\underline{J}_3^{+*} \cdot \underline{E}_3)$$

where $\underline{J}_3^{+*}$ is the complex conjugate of $\underline{J}_3^+$. By substituting the results from equations (2.100) and (2.112) into this expression, it may be shown that

$$W_3^i = \frac{\omega_{p+}^2}{2\pi\Omega_+} \Big\{ Re(M_{3xx}^+)(|E_{3x}|^2 + |E_{3y}|^2 - i\,Im(M_{3xy}^+)(E_{3x}E_{3y}^* - E_{3x}^* E_{3y})$$

$$+ Re(M_{3zz}^+)|E_{3z}|^2 + i\,Im(M_{3yz}^+)(E_{3z}E_{3y}^* - E_{3z}^* E_{3y})$$

$$+ 2Re(M_{3xz}^+) Re(E_{3x}E_{3z}^*) \Big\} \qquad \ldots(2.113)$$

where $\qquad M_{3xy}^+ = Re(M_{3xy}^+) + i\,Im(M_{3xy}^+)$

Similarly, the electron energy absorption from the mixed wave may be shown to be

$$W_3^e = \text{Real}\ (\underline{J}_3^{-*} \cdot \underline{E}_3)$$

$$= -\frac{\omega_{p-}^2}{2\pi\Omega_-} \Big\{ Re(M_{3xx}^-)(|E_{3x}|^2 + |E_{3y}|^2) - i\,Im(M_{3xy}^-)(E_{3x}E_{3y}^* - E_{3x}^* E_{3y})$$

$$+ Re(M_{3zz}^-)|E_{3z}|^2 + i\,Im(M_{3yz}^-)(E_{3z}E_{3y}^* - E_{3z}^* E_{3y})$$

$$+ 2Re(M_{3xz}^-) Re(E_{3x}E_{3z}^*) \Big\} \qquad \ldots(2.114)$$

Equation (2.113) gives the total ion energy absorption, that is, contributions from both collisionless and collisional damping. By the use of equation (2.100), the contributions from cyclotron damping to W_3^i may be shown to be

$$W_3^i\,(\text{collisionless}) = \frac{\omega_{p+}^2}{4\sqrt{\pi}\,K_z}\sqrt{\frac{m_+}{2KT^+}}\;e^{-(\alpha_{-1}^+)^2}\left\{\left|E_{3x}+iE_{3y}\right|^2 + \frac{\Delta K_x}{K_z}\,\mathrm{Re}\left[E_{3z}^*\left(E_{3x}+iE_{3y}\right)\right] + \left(\frac{\Delta K_x}{K_z}\right)^2\left|E_{3z}\right|^2\right\} \qquad \ldots(2.115)$$

where

$$(\alpha_{-1}^+) = \frac{(\omega-\Omega_+)}{K_z}\sqrt{\frac{m_+}{2KT^+}}$$

$$\omega = (1+\Delta)\,\Omega_+$$

CHAPTER 3 CALCULATIONS FOR THE PROPOSED EXPERIMENT

In this chapter, experimental considerations such as the selection of the D-C magnetic field (Bo), the choice for the incident frequencies ω_1 and ω_2, and the resulting sensitivities to frequency, density, and angular disturbances will be discussed. Magnetized plasmas with densities in the range $10^{12}/cm^3$ to $10^{15}/cm^3$ will be considered.

3.1 Selection of the D-C Magnetic Field

The incident waves in the plasma will be only slightly Landau damped by the electrons if the phase velocities of the incident waves in the direction of the static magnetic field are much greater than the electron thermal velocity. The role of the collisionless damping effects on the incident waves may be examined more closely by writing equation (2.78) for the incident waves.

$$F_o^{\pm} = \sqrt{\pi}\,\frac{K_{iz}}{|K_{iz}|}\,\exp -\left(\alpha^{\pm}_{i(n)}\right)^2 + \frac{i}{\alpha^{\pm}_{i(n)}}\left(1 + \frac{1}{2\left(\alpha^{\pm}_{i(n)}\right)^2} + \cdots\right) \qquad \ldots(3.1)$$

where $$\alpha^{\pm}_{i(n)} = \frac{(\omega_i + n\Omega_{\pm})}{K_{iz}}\sqrt{\frac{m_{\pm}}{2KT^{\pm}}}$$

$i = 1,2$ and denotes the first and second incident waves respectively.

In the evaluation of the dispersion relations for the incident waves, mobility tensors similar to that given by equation (2.99) for

the difference frequency wave, may be defined. The various components of this tensor will be expressed in terms of F_o, F_1, F_2.. (F_1, F_2.. may be expressed in terms of F_o by the use of equation (2.69) and (2.70)). The first term on the RHS of equation (3.1) gives the collisionless damping effects, while the first component in the second term gives the normal cold plasma contribution. Only the latter term was used in evaluating the mobility tensor for the incident waves in Section (2.2.1) (see equation (2.12)). It may be shown from equation (3.1), that the effects of collisionless damping on the incident waves will be small if $|\alpha^{\pm}_{i(n)}| \gg 1$. The effects of electron Landau damping are represented by n = 0, where $\alpha^{-}_{i(o)}$ is just the ratio of the phase velocity of the incident wave in the plasma to the electron thermal velocity. The case where $n \neq 0$ represents the contribution of the n'th harmonic of cyclotron damping.

If $\alpha^{\pm}_{i(o)}$ is of the order of 3, then only about one particle in 2400 will have a velocity greater than or equal to the phase velocity of the incident wave[27]. By choosing $|\alpha^{\pm}_{i(n)}| \gtrsim 5$, the effects of collisionless damping on the incident waves can be regarded as negligible. Therefore the static magnetic field must be selected such that the dispersion relations for the incident waves in the plasma satisfy the condition that $|\alpha^{\pm}_{i(n)}| \gtrsim 5$. This would then insure that the Landau and cyclotron damping of the incident waves is negligible.

In order that the static magnetic field (B_o) may be selected such that $|\alpha^{-}_{i(n)}| \gtrsim 5$, an estimate for the maximum electron temperature that can be realized in an experiment is required. This will now be calculated.

If ion effect are neglected, the ratio of the collisionless to the collisional damping of the incident waves may be shown to be equal to

$$\frac{(\omega_i + n\Omega_-)}{\nu_{ei}} \alpha^-_{i(n)} \sqrt{\pi} \; \exp - (\alpha^-_{i(n)})^2 \qquad \ldots(3.2)$$

For the case in which $|\alpha^-_{i(n)}| \geq 5$

$$\frac{\text{collisionless damping}}{\text{collisional damping}} \leq 3.9 \times 10^{-12} \; \frac{(\omega_i + n\Omega_-)}{n_0} T^{3/2}$$

The above ratio does not exceed one for plasmas with densitites greater than $10^{12}/cm^3$, an electron temperature less than 5×10^7 °K, and where the frequencies of the incident waves are below the infrared region.

By choosing $|\alpha^-_{i(n)}|$ to be at least 5, the electrons will accept energy from the incident fields essentially through collisional damping. The collisional heating of electrons by the second order fields will be neglected since these fields are much smaller than the electric fields associated with the incident waves in the cases considered. The maximum electron temperature that can be realized in an experiment is bounded by one of two factors. These are:

(i) Since the dependence of the electron-ion collision frequency upon temperature is proportional to $T^{-3/2}$, the rate at which electrons are heated by the incident fields will decrease with increasing temperature. The maximum electron temperature that can be realized in an experiment will be that for which the energy radiated by the electrons through Bremsstrahlung and cyclotron radiation equals the energy absorbed by the electrons through the collisional damping of the incident waves.

(ii) The maximum electron temperature that can be realized in an experiment will also be bounded by the finite time that a plasma may be confined. That is, if the plasma is confined for only one milli-second, the electron temperature may not reach the limit given by condition (i) above, where the rate of energy absorption by the electrons through the collisional damping of the incident waves equals the energy radiated through bremsstrahlung and cyclotron radiation.

By performing a cold plasma analysis on the incident waves, the power absorbed by the electrons from the incident waves may be shown to be

$$W_1^e = \frac{\omega_1 \nu \omega_{p-}^2}{4\pi} \left\{ \frac{(\omega_1 - \Omega_-)}{(\omega_1 + \Omega_-)^2 (\omega_1 - \Omega_+)^2 + \omega_1^2 \nu^2} \left| E_{1x} + i E_{1y} \right|^2 \right.$$

$$\left. + \frac{(\omega_1 + \Omega_+)}{(\omega_1 - \Omega_-)^2 (\omega_1 + \Omega_+)^2 + \omega_1^2 \nu^2} \left| E_{1x} - i E_{1y} \right|^2 + \frac{1}{\omega_1^3} \left| E_{1z} \right|^2 \right\} \text{ ergs/cm}^3\text{/sec.} \qquad \ldots(3.3)$$

$$\nu = \nu_{ei} \, (1 + m_-/m_+)$$

ν_{ei} = electon-ion collision frequency $\sim \dfrac{31.7 n_o}{T^{3/2}}$ (see Clavier[28])

Since only a RH elliptically polarized wave is transmitted into the plasma, the expression for W_1^e given by equation (3.3) may be approximated as follows, if terms of the order (Ω_+/ω_1) are neglected:

$$W_1^e \sim \frac{\omega_{p-}^2 \nu}{\pi (\omega_1 - \Omega_-)^2} \left| E_{1x} \right|^2 \text{ ergs/cm}^3\text{/sec.} \qquad \ldots(3.4)$$

If for the moment, the effects of bremsstrahlung and cyclotron

radiation are neglected, the resulting ordinary differential equation governing the electron temperature during the heating cycle is given by

$$\frac{d}{dt}\left(\frac{3}{2} nKT^-\right) = \frac{40.3\,\omega_{p-}^2}{(\omega_1-\Omega_-)^2}\,\frac{n_o}{(T^-)^{3/2}}\,|E_{1x}|^2 \qquad \ldots(3.5)$$

where the energy gained from each of the two incident waves is taken to be approximately equal. In writing the above equation, it is assumed that the heating that occurs in the time, $2\pi/\omega_{1(2)}$, is small. Therefore, the time average of the electron heating by the incident waves appears on the RHS of equation (3.5). If the initial plasma temperature is approximated as zero,

$$(T_f^-)^{5/2} = 4.87\times 10^{17}\,\frac{\omega_{p-}^2}{(\omega_1-\Omega_-)^2}\,|E_{1x}|^2 \times \text{heating period (sec.)} \qquad \ldots(3.6)$$

where T_f^- = electron temperature at the end of the heating period. An estimate for the value of $w_{p-}^2/(w_1 - \Omega_-)^2$ is required if equation (3.6) is to be used for finding the maximum electron temperature. This information may be obtained from the conditions that $|\alpha_{i(o)}^{-}|$ and $|\alpha_{i(-1)}^{-}|$ must be greater than or equal to five if the electron Landau and cyclotron damping of the incident waves in the plasma is to be negligible. The ratio of the two parameters is given by

$$\frac{|\alpha_{1(0)}^-|}{|\alpha_{1(-1)}^-|} = \frac{C_1}{|C_1 - 42.8|}$$

$$\begin{aligned} &> 1 \quad \text{for} \quad C_1 > 21.4 \\ &< 1 \quad \text{for} \quad C_1 < 21.4 \end{aligned} \qquad \ldots(3.7)$$

where $w_1 = C_1 \Omega_H = C_1 \sqrt{\Omega_+ \Omega_-}$

If the condition that both $|\alpha_{1(o)}|$ and $|\alpha^-_{1(-1)}|$ be greater than five is to be satisfied, the static magnetic field (B_o) must be chosen such that the following conditions are satisfied.

$$
\begin{aligned}
|\alpha^-_{1(-1)}| &\geq 5 \quad \text{for} \quad \frac{\Omega_-}{2} \leq \omega_1 < \Omega_- \\
|\alpha^-_{1(0)}| &\geq 5 \quad \text{for} \quad \Omega_+ < \omega_1 \leq \frac{\Omega_+}{2}
\end{aligned}
\qquad \ldots(3.8)
$$

For instance, for $\Omega_+ < w_1 < \Omega_-/2$, if $\alpha^-_{1(o)} \geq 5$, then by equation (3.7) $\alpha^-_{1(o)} / \alpha^-_{1(-1)} < 1$, so that $\alpha^-_{1(1)}$ is also greater than or equal to 5.

The use of the dispersion relation for the first incident wave in equation (3.8) yields

(i) $\Omega_+ < \omega_1 \leq \Omega_-/2$

$$\frac{\omega_{p-}^2}{(\omega_1 - \Omega_-)^2} \ll \frac{(C_1 + 1/42.8)}{(C_1 - 42.8)} \left(1 - \frac{1.19 \times 10^8}{T^-_{max}}\right)$$

(ii) $\frac{\Omega_-}{2} \leq \omega_1 < \Omega_-$ $\qquad \ldots(3.9)$

$$\frac{\omega_{p-}^2}{(\omega_1 - \Omega_-)^2} \leq \frac{(C_1 + 1/42.8)}{(C_1 - 42.8)} \left(1 - \frac{(C_1 - 42.8)^2}{C_1^2} \frac{1.19 \times 10^8}{T^-_{max}}\right)$$

The bound on maximum electron temperature, resulting from the finite time a plasma may be confined, is obtained by using the results of equation (3.9) in equation (3.6).

If terms of the order $(T^-_{max}/1.2 \times 10^8)$ are neglected, the following estimates for the maximum electron temperature may be made.

(i) $\Omega_+ < \omega_1 \leq \Omega_-/2$

$$T^-_{MAX} \leq \left[\frac{3.64 \times 10^{26}}{25} \frac{(C_1 + 0.023)}{(42.8 - C_1)} \left|E^V_{inc}\right|^2 \times \text{heating period (sec)} \right]^{2/7} \text{°Kelvin} \qquad \ldots(3.10)$$

(ii) $\Omega_-/2 \leq \omega_1 < \Omega_-$

$$T^-_{MAX} \leq \left[\frac{3.64 \times 10^{26}}{25} \frac{(C_1 + 0.023)(42.8 - C_1)}{C_1^2} \left|E^V_{inc}\right|^2 \times \text{heating period (sec)} \right]^{2/7} \text{°Kelvin}$$

where E^V_{inc} = maximum value of the incident electric field in stat-volts/cm. (Note that the incident fields are assumed to be RH circularly polarized).

The bound on the maximum electron temperature resulting from a balance between the energy absorbed by the electrons from the incident waves and the energy radiated through bremsstrahlung and cyclotron radiation at the maximum electron temperature will now be developed. The relations for the energy radiated through bremsstrahlung and cyclotron radiation are taken from Rose and Clark[3].

$$W_b = \text{bremsstrahlung radiation} = 1.41 \times 10^{-27}\, Z^2 n_+ n_- (T^-)^{1/2} \text{ ergs/cm}^3\text{/sec} \qquad \ldots(3.11)$$

$$W_c = \text{cyclotron radiation} = 5.35 \times 10^{-25}\, B_o^2\, n_-\, T^- \left(1 + \frac{T^-}{2.4 \times 10^{10}} + \ldots\right)$$

ergs/cm^3/sec.

where B_o = magnetic field in gauss

n_+ = ion density/cm^3

n_- = electron density/cm^3

T^- = electron temperature in degrees Kelvin.

For the large values of static magnetic field being considered, electron cyclotron radiation will dominate over bremsstrahlung radiation. A balance is obtained by equating W_c to W_1^e. With the help of equation (3.9), this gives the following limits on the maximum electron temperature:

(i) $\Omega_+ < \omega_1 \leq \Omega_-/2$

$$T^-_{MAX} \leq \left[1.59 \times 10^{44} \frac{C_1^2}{n_o} \left(E^V_{inc}\right)^2 \right]^{2/9} \ {}^\circ\text{Kelvin}$$

...(3.12)

(ii) $\frac{\Omega_-}{2} \leq \omega_1 < \Omega_-$

$$T^-_{MAX} \leq \left[1.59 \times 10^{44} \frac{(C_1 - 42.8)^2}{C_1^2 n_o} \left(E^V_{inc}\right)^2 \right]^{2/9} \ {}^\circ\text{Kelvin}$$

In deriving the expressions for T^-_{max} in equations (3.10) and (3.12), the reflection of the incident waves at the plasma-vacuum interface has been neglected. This would imply, that in an experiment, the actual value of T^-_{max} will be slightly less than that given by these equations. This will result in the value for the D-C magnetic field (B_o) being slightly larger than is required for satisfying the conditions given by equation (3.8).

Equations (3.9), (3.10) and (3.12) may now be used to give the required value for B_o, so that the collisionless damping of the incident waves in the plasma is negligible.

(i) $\Omega_+ < \omega_1 \leq \Omega_-/2$

$$B_o^2 \geq \frac{1.88 \times 10^{-2}\, n_o}{\left(C_1 + \frac{1}{42.8}\right)\left(C_1 - 42.8\right)\left(1 - \frac{2.97\times 10^9}{25\, T^-_{MAX}}\right)} \qquad \ldots(3.13)$$

(ii) $\Omega_-/2 \leq \omega_1 < \Omega_-$

$$B_o^2 \geq \frac{1.88 \times 10^{-2}\, n_o}{\left(C_1 + \frac{1}{42.8}\right)\left(C_1 - 42.8\right)\left[1 - \frac{(C_1 - 42.8)^2}{C_1^2}\,\frac{2.97\times 10^9}{25\, T^-_{MAX}}\right]}$$

The value for T^-_{max} that is to be substituted into the expressions for B_o^2 in equation (3.13) is the minimum of the values obtained for T^-_{max} in equations (3.10) and (3.12). It should be noted that the equation for B_o^2 when $\Omega_-/2 \leq \omega_1 < \Omega_-$, places an upper bound on ω_1, namely that $\frac{(c_1 - 42.8)^2}{c_1^2}\,\frac{2.97 \times 10^9}{25\, T^-_{max}} > 1$. That is, for ω_1 sufficiently close to Ω_-, it will not be possible to satisfy the condition that $|\alpha^-_{1(-1)}| \geq 5$ by increasing B_o, since the phase velocities for the incident waves in the plasma are less than or equal to the speed of light.

3.2 Techniques for Optimizing Ion energy Absorption

Three techniques are available for optimizing the energy absorbed by the ions from the second order fields. These are

(i) By allowing the difference frequency, ω, to approach the ion cyclotron frequency, a resonance in the second order ion current is obtained. The peak in this resonance is bounded by

either collisional or cyclotron damping effects, depending upon which process is dominant. This technique places restrictions on the allowable frequency and D-C magnetic field fluctuations. The effect of density fluctuations is small.

(ii) Another technique for optimizing the ion energy absorption from the second order fields is to obtain a resonance in the magnitude of the second order fields, by allowing the mixed wave to approach a natural mode in the plasma. If ω is sufficiently removed from the ion cyclotron frequency, the resonance will be limited by collisional effects. This method however imposes stringent restrictions on the allowable density fluctuations and the allowable perturbations in the angle of incidence for the sources (see Figure (2.2)), in addition to restrictions on frequency and static magnetic field disturbances.

(iii) A third possible scheme is the combination of the above two techniques. However, because a field resonance effect is utilized, this method of ion heating will be very sensitive to density, frequency, static magnetic field, and angular disturbances.

Examples of the first and third schemes will be given in section (3.4). Examples of the second scheme have been given by James and Thompson[17], and Jayasimha[18]. These will be discussed in Chapter 4.

3.3 Anisotropic Pressure Effects

When the first and third schemes in Section (3.2) are used,

the energy from the mixed wave is selectively coupled to the ions in a direction perpendicular to the static magnetic field. Rapid randomization of the ordered motion perpendicular to the magnetic field will then occur through fine scale mixing as described by Berger et al[1] and Stix[29], increasing the temperature of the transverse motion of the ions. That is, the thermal motion of the ions in the z-direction will cause ions which are in one region of the wave to move into another region where the phase of the wave is different. A large number of such transitions will tend to increase the temperature of the transverse ion motion. Once the phase mixing has occured, collisional effects will tend to return the plasma to Maxwellian. However, the maximum perpendicular ion temperature ($T_\perp^+$) that may be realized is bounded by the ion temperature parallel to the magnetic field ($T_\parallel^+$) by the onset of an ion cyclotron overstability. The limit on $T_\perp^+$ is approximately given by

$$\frac{T_\perp^+}{T_\parallel^+} < \beta_\parallel^{-1/3} \qquad \ldots(3.14)$$

where $$\beta_\parallel = \frac{8\pi n K T_\parallel^+}{B_o^2} = \frac{\text{ion kinetic pressure parallel to } B_o}{\text{magnetic pressure}}$$

For the examples to be considered in the next section, the ratio of $T_\perp^+/T_\parallel^+$ cannot exceed twenty. This is however a pessimistic estimate for the maximum ion temperature because the ion temperature parallel to the magnetic field will increase during the heating cycle through collisional effects. To extend the upper limit on the ion temperature as given by equation (3.14), the plasma may be heated by a sequence of pulses. The period between the pulses could then be arranged to be

of sufficient length so that the ions reach equilibrium before each heating pulse.

The power absorbed by the ions through the collisionless damping of the second order fields for the case where $T_\perp^+ \neq T_\parallel^+$ may be shown to be

$$W_3^i \,(\text{collisionless}) = \frac{\omega_{p+}^2 K T_\perp}{2\sqrt{\pi}\; K_z\, m_+} \left(\frac{m_+}{2KT_\parallel^+}\right)^{3/2} \left[\frac{(\omega-\Omega_+)T_\perp + \Omega_+ T_\parallel}{\omega T_\perp}\right] \exp -(\alpha_{-1}^+)^2 \qquad \ldots(3.15)$$

$$\left\{ \left|E_{3x} + iE_{3y}\right|^2 + \frac{(\omega-\Omega_+)^2}{\Omega_+^2}\left(\frac{K_x}{K_z}\right)^2 \left|E_{3z}\right|^2 + \frac{2(\omega-\Omega_+)}{\Omega_+}\frac{K_x}{K_z}\,\mathrm{Re}\left(E_{3z}^*\,(E_{3x}+iE_{3y})\right)\right\}$$

Equation (2.115) may be obtained by setting $T_\perp^+ = T_\parallel^+$ in equation (3.15).

If $T_\parallel^+$ is taken to be constant, and if ω is sufficiently close to Ω_+, it may be seen from equation (3.15) that the power absorbed by the ions through the cyclotron damping of the difference frequency wave will remain essentially constant with increasing $T_\perp^+$. From the above equation it may also be seen that the ions will absorb energy from the driven second order fields only if

$$\left[\frac{(\omega-\Omega_+)T_\perp + \Omega_+ T_\parallel}{\omega T_\perp}\right] > 0 \qquad \ldots(3.16)$$

This requires that $\quad \Delta > -T_\parallel^+/T_\perp^+ \qquad$ where $\omega = (1+\Delta)\,\Omega_+$

By using the limit established by the condition of a cyclotron overstability for $T_\parallel^+/T_\perp^+$ (equation (3.14), the restriction on Δ in equation (3.16) becomes

$$\Delta > -\beta_\parallel^{1/3} \qquad \ldots(3.17)$$

3.4 Examples

In this section, the theory developed in Chapter 2 and in Sections (3.1) and (3.2) of this chapter, will be applied to plasmas with densities in the range $10^{12}/cm^3$ to $10^{15}/cm^3$. For the cases considered, the square of the ratio of the ion Larmor radius to the perpendicular wavelength will have a value less than 0.5 for a maximum ion temperature of 10^8 degrees Kelvin. This ratio will be represented by λ^+. In the examples considered, the incident wave outside the plasma will be taken to be RH circularly polarized, with an electric field of magnitude of E^v_{inc}. This paricular wave is chosen for better matching at the plasma-vacuum interface, since the transmitted wave is almost RH circularly polarized. The numerical calculations were performed on an IBM 360 computer.

In Figure (3.1), the static magnetic field (B_o) and the value for W^i_3 maximized with respect to the angle of incidence θ^s_1 (see Figure (2.2)) for each C_1 without the use of field resonance effects, have been plotted as a function of C_1, where $\omega_1 = C_1\sqrt{\Omega_+\Omega_-} = C_1\Omega_H$. It is evident from this Figure that a good choice for the frequency of the incident waves is in the neighborhood of the hybrid frequency. If the static magnetic field is at a premium, a higher incident frequency may be used. However, unless a field resonance effect is utilized, this will result in a reduction in the rate of ion energy absorption.

Figure (3.2) shows the dependence of W^i_3 on the angle of incidence

for the first source (see Figure (2.2)), with ω_1 being equal to 1.1 times the hybrid frequency. The sharp peak in W_3^i for $\theta_1^s \sim 35°$ is the result of a field resonance effect. This resonance is strongly dependent upon density, angular, frequency and magnetic field fluctuations as is shown by Figures (3.3), (3.4), and (3.5). If the sensitivities are defined using the one-quarter power points as in James and Thompson[17], then the following sensitivities may be defined for the above field resonance.

$$\begin{aligned} &\delta\omega_1/\omega_1 \sim .67 \times 10^{-5} \\ &\delta n_o/n_o \sim 7 \times 10^{-4} \\ &\delta B_o/B_o \sim 3.1 \times 10^{-4} \\ &\delta(\theta_1^s) \sim 10^{-4} \text{ radians} \\ &\text{for } T = 10^6\ °K \end{aligned} \qquad \ldots(3.18)$$

The dependence of W_3^i on the detuning of the difference frequency from the ion cyclotron frequency and the effects of density and static magnetic field perturbations is given in Figures (3.6) and (3.7) for $\theta_1^s = 64.1°$. If the ions are to gain energy through cyclotron damping, the above choice for an operating point represents the worst case for sensitivity to frequency and static magnetic field fluctuations. The value for K_z in this case is just large enough so that cyclotron damping occurs for $(\omega - \Omega_+)/\Omega_+ \sim 10^{-5}$. That is $|\alpha_{-1}^+| = \left|\frac{(\omega - \Omega_+)}{K_z}\sqrt{\frac{m_+}{2KT^+}}\right| \sim 1$ for $(\omega - \Omega_+)/\Omega_+ \sim 10^{-5}$. If ω is detuned any further from Ω_+, the difference frequency wave will not be cyclotron damped. This is the reason for the high sensitivity to frequency and static magnetic field

fluctuations at this operating point.

A much better choice for an operating point is given by $\theta_1^s = 50.2°$. The results for this case are given by Figures (3.8) and (3.9). In this case, the absolute value of K_z is much larger than that obtained when $\theta_1^s = 64.1°$. This may be seen by examining the values for $|\alpha_o^-|$ ($\alpha_o^- = \omega/K_z \sqrt{\frac{m_-}{2KT^-}}$) in Figure (3.10) for the case where the slope of the plasma-vacuum interface is zero, and θ_1^s has the values 50.2° and 64.1°. This would imply that for the operating point given by $\theta_1^s = 50.2°$, the resulting sensitivities to frequency and static magnetic field fluctuations will not be as critical as in the case where $\theta_1^s = 64.1°$. The sensitivities obtained from the quarter power points are as follows for the case where $\theta_1^s = 50.2°$.

$$
\begin{aligned}
&\delta\omega_1/\omega_1 \sim 1.2 \times 10^{-5} \\
&\delta n_o/n_o > .2 \\
&\delta B_o/B_o \sim 5.6 \times 10^{-4} \\
&\delta(\theta_1^s) < 30° \text{ radians} \\
&\text{for } T = 10^6 \text{ °K}
\end{aligned}
\qquad \ldots(3.19)
$$

By using the results in Figure (3.10), it may be shown that plus and minus five degree fluctuations in the slope of the plasma vacuum interface may be tolerated without leading to excessive electron Landau damping of the mixed wave. From Figure (3.6) and (3.8), it is evident that with increasing temperature, the rate at which energy is absorbed by the ions will drop slightly, however an improvement will be obtained

in the frequency bandwidth over which the mixed wave is cyclotron damped.

The field resonance for $\omega_1 = 10\Omega_H$ is investigated in Figures (3.12) to (3.14). If the sensitivities are defined by the one-quarter power points as in equation (3.11), then for the above resonance

$$\begin{aligned} &\delta\omega_1/\omega_1 \sim 9.1 \times 10^{-7} \\ &\delta n_o/n_o \sim 7.4 \times 10^{-4} \\ &\delta B_o/B_o \sim 3.9 \times 10^{-4} \\ &\delta(\theta_1^s) \leq 1.6 \times 10^{-5} \text{ radians} \\ &\text{for } T = 10^6 \text{ °K} \end{aligned} \qquad \ldots(3.20)$$

The field resonance for $\omega_1 = 10\Omega_H$ is considerably "sharper" than that for $\omega_1 = 1.1\Omega_H$, resulting in tighter restrictions for the allowable frequency and angular disturbances on the incident waves.

The results for a plasma with a density of $10^{15}/cm^3$ is given in Figures (3.15) to (3.21).

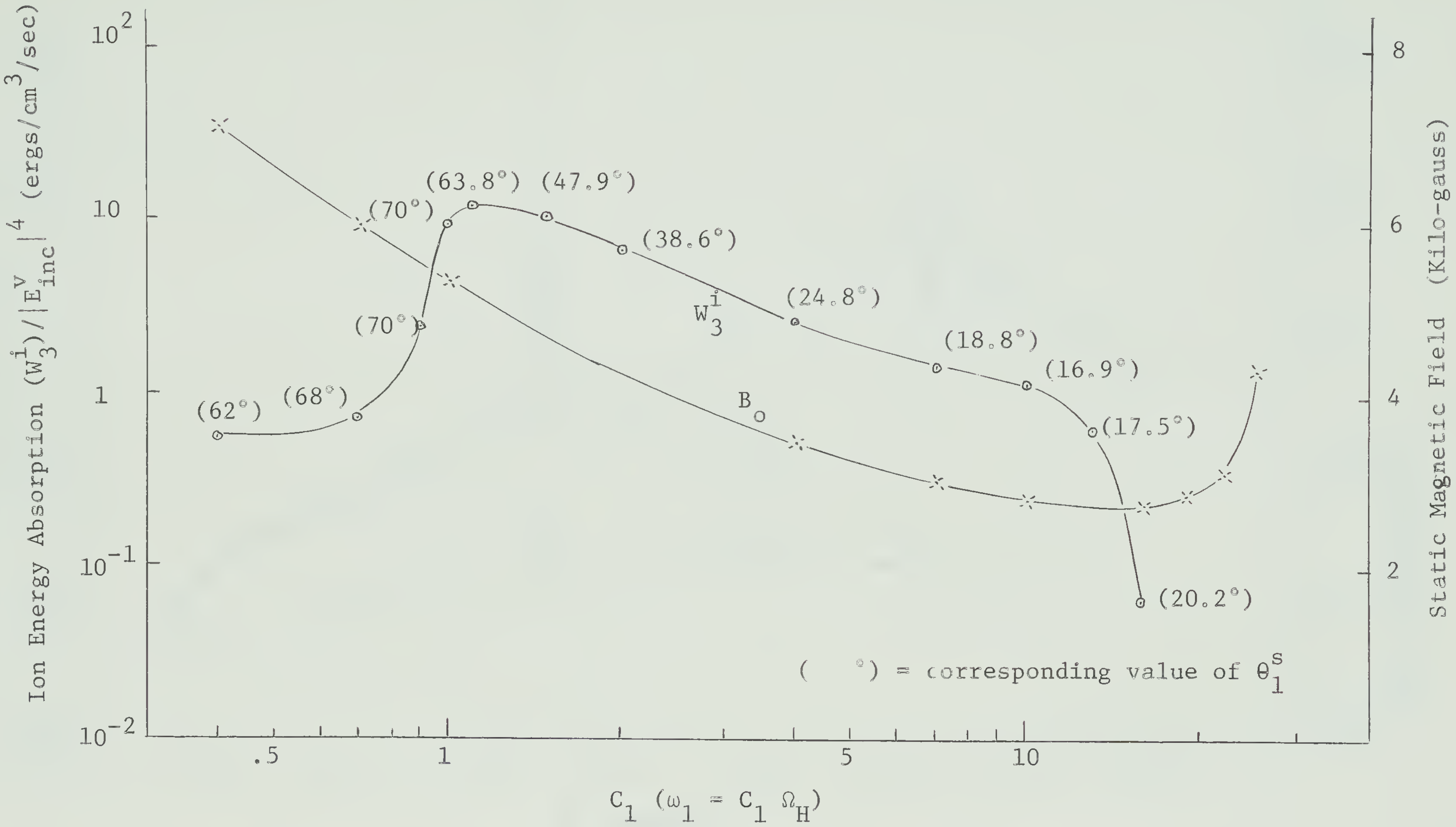

Figure 3.1. Ion energy absorption and B_o as a Function of C_1. The physical parameters are: plasma density = 10^{12}/cm^3, temperature = 10^6 °K, $|E_{inc}^v| \leq 15$ KV/cm, $\lambda_{max}^+ \leq 0.5$ for $T_{max}^+ = 10^8$ °K, plasma confined for 10^{-3} sec., and $(\omega - \Omega_+)/\Omega_+ = 10^{-5}$. The d-c magnetic field is selected per equation (3.13).

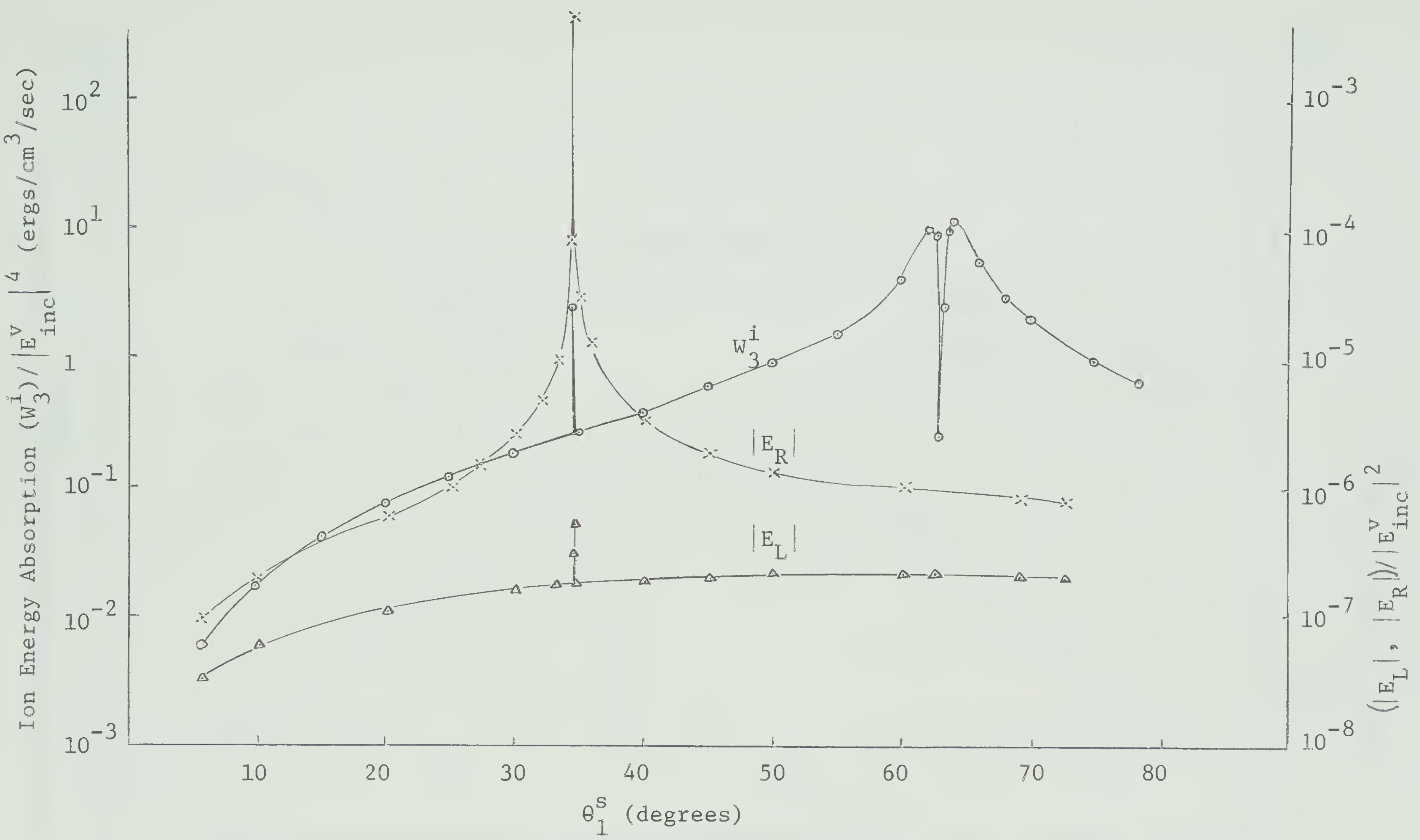

Figure 3.2. Ion Energy Absorption and the Second Order Electric Field as Functions of the Angle of Incidence for the First Source. The physical parameters are as given in Figure (3.1), with $C_1 = 1.1$, $B_o = 5.2$ Kg, and $\omega_1 = 3.7 \times 10^8$ Hertz. $E_L = E_{3x} + iE_{3y}$, $E_R = E_{3x} - iE_{3y}$

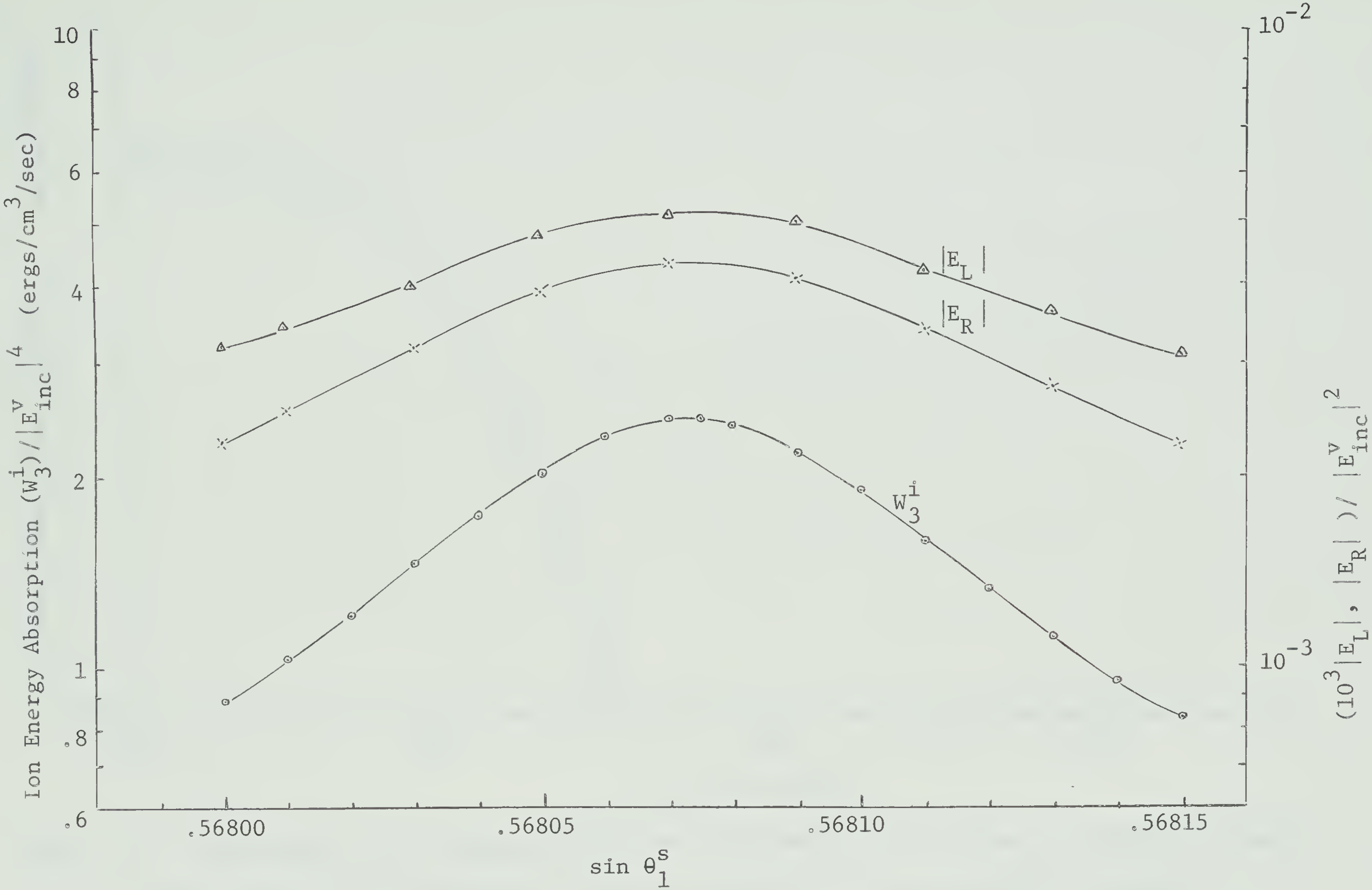

Figure 3.3. Investigation of Field Resonance as a Function of the Angle of Incidence for the First Source. The physical parameters are as given in Figure (3.2).

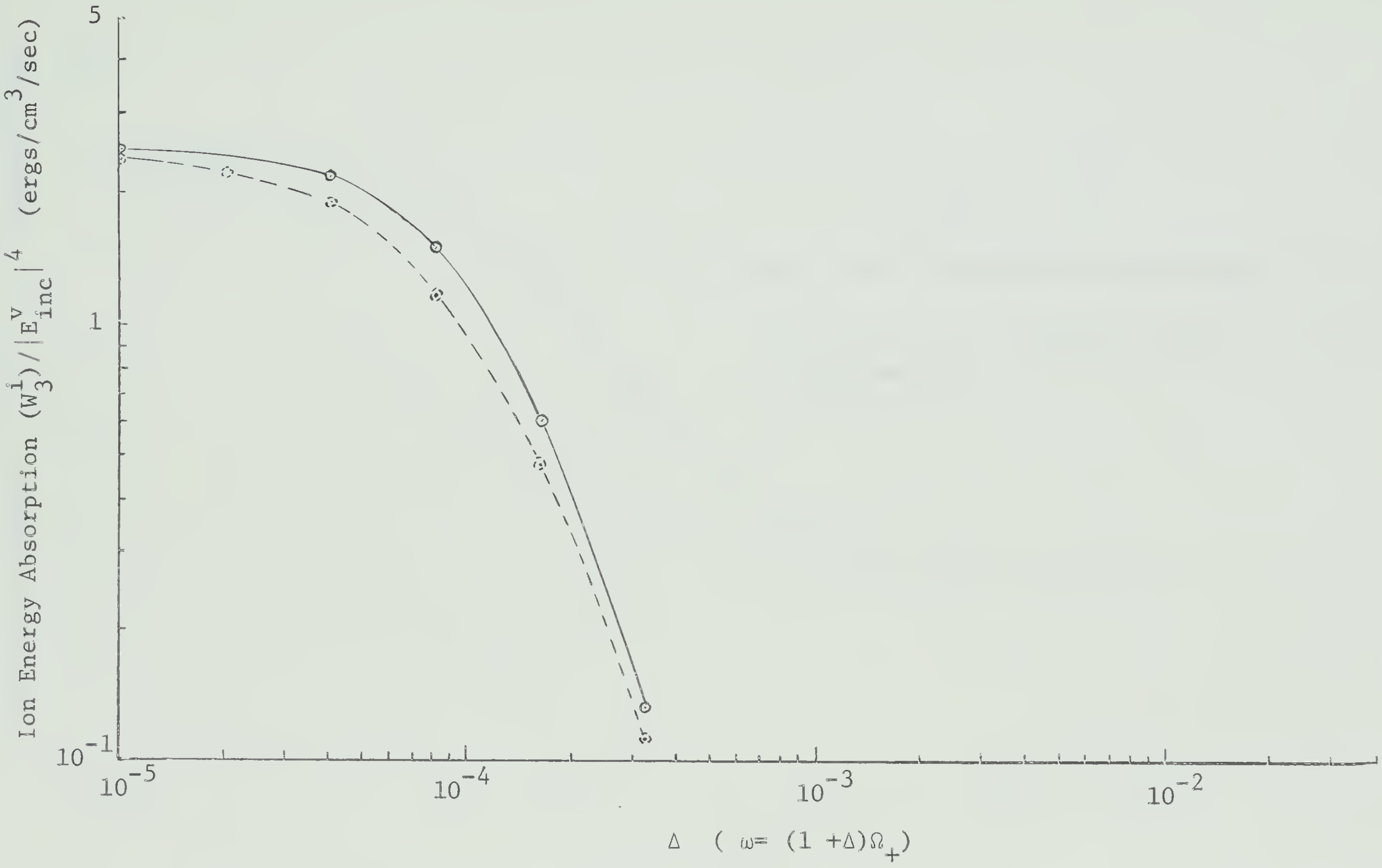

Figure 3.4. Sensitivity of ion energy absorption to frequency fluctuations at a field resonance point. The physical parameters are as in Figure (3.3), with $\sin(\theta_1^s) = 0.568073$. The results for negative Δ are given by the broken curve.

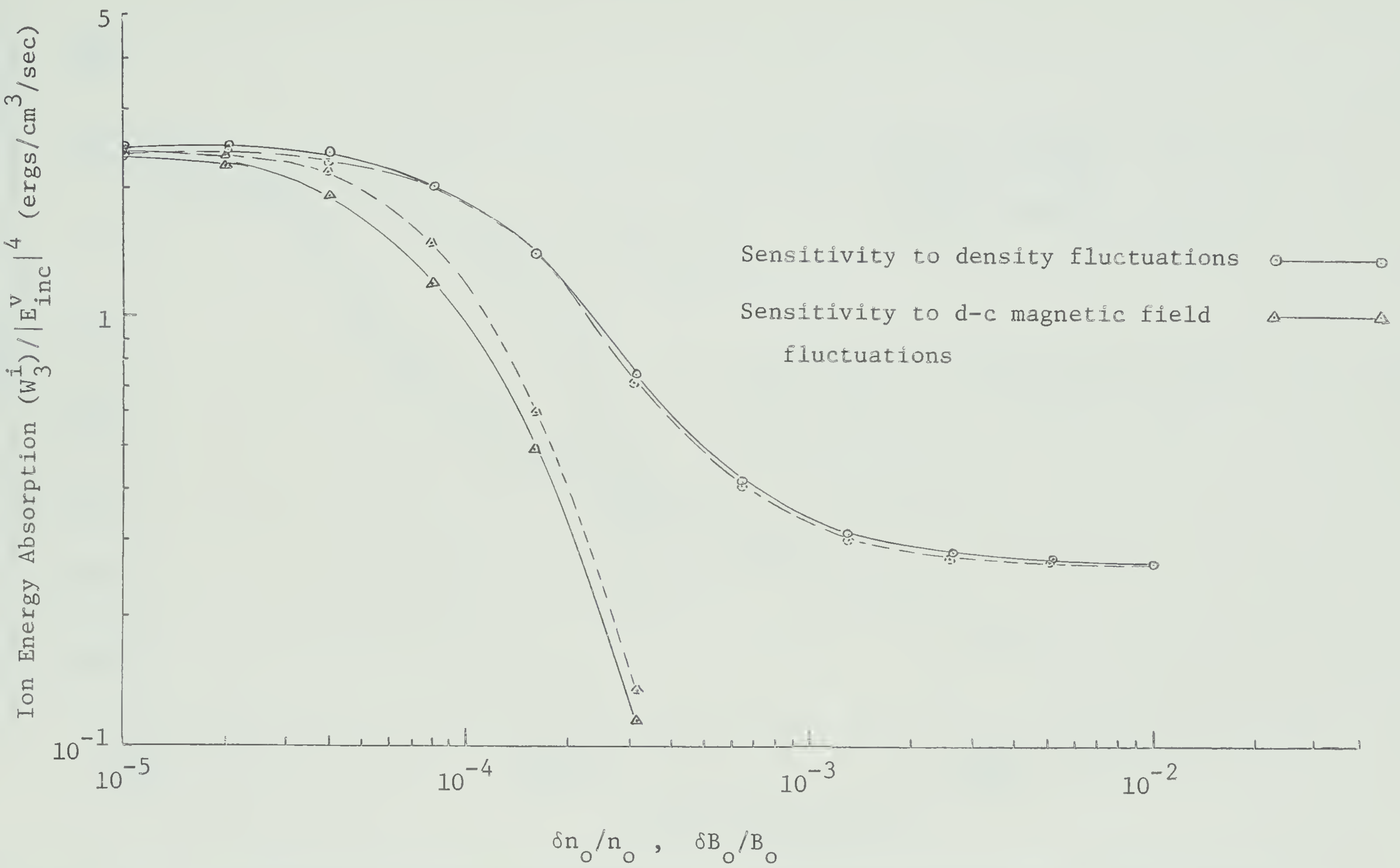

Figure 3.5. Sensitivity of ion energy absorption to density and d-c magnetic field fluctuations. The physical parameters are as in Figure (3.3), with $\sin(\theta_1^s) = 0.568073$. The results for negative δn_o and δB_o are given by the broken curves.

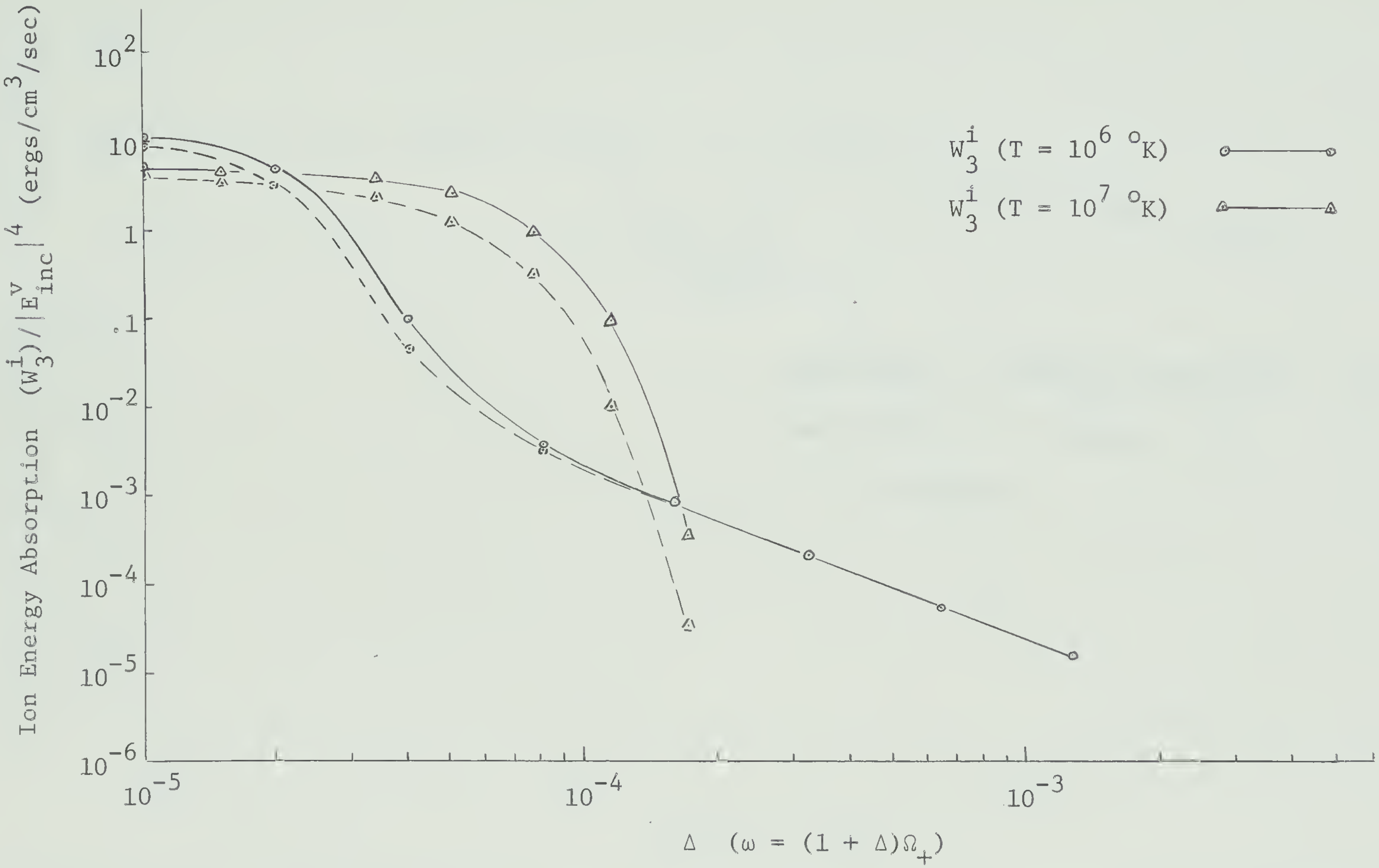

Figure 3.6. Sensitivity of ion energy absorption to frequency fluctuations. The physical parameters are as given in Figure (3.2), with $\theta_1^s = 64.1^o$. The results for negative Δ are given by the broken curves.

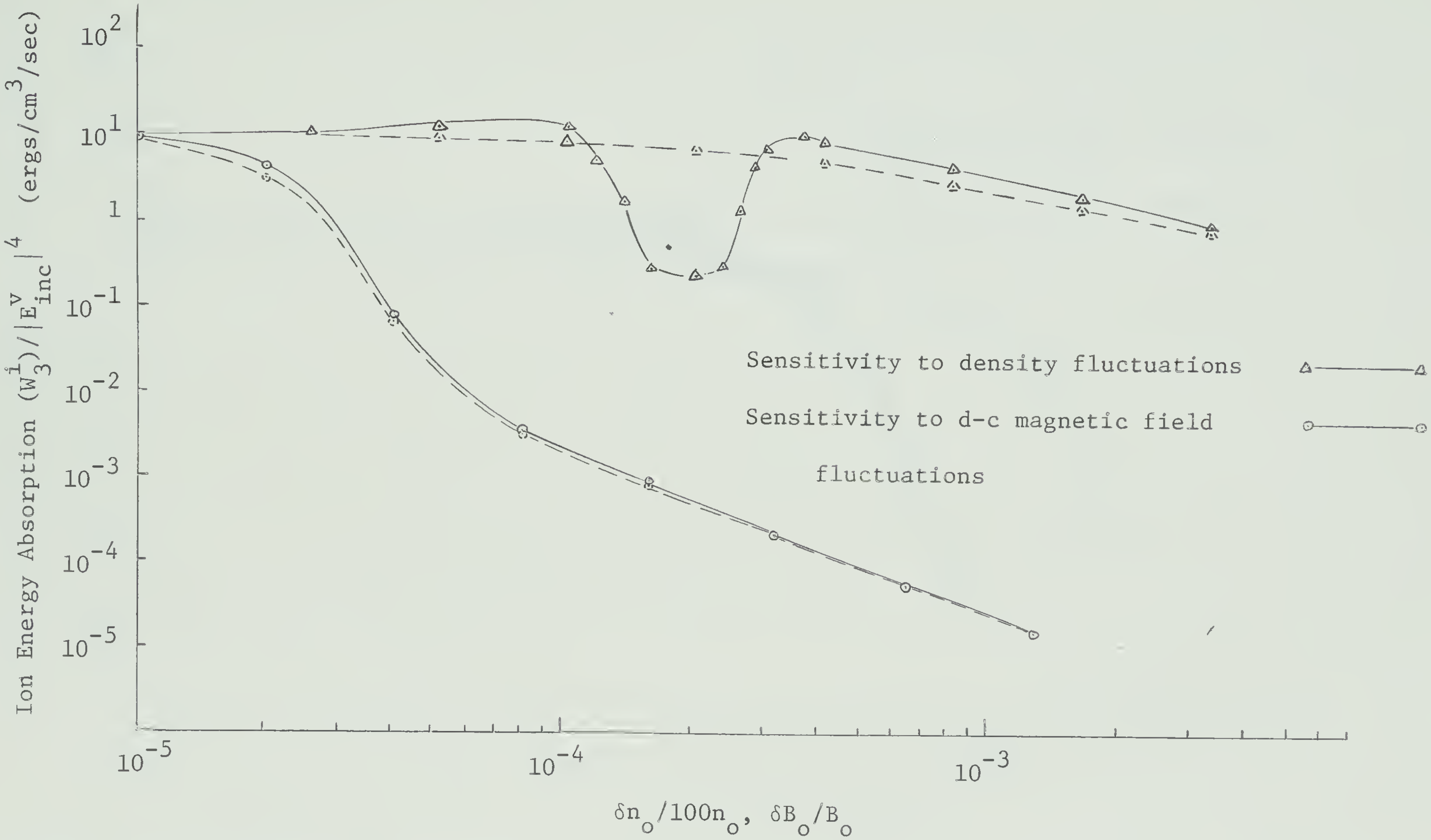

Figure 3.7. Sensitivity of ion energy absorption to density and d-c magnetic field fluctuations. The physical parameters are as given in Figure (3.2), with $\theta_1^s = 64.1°$. The results for negative δn_o and δB_o are given by the broken curves.

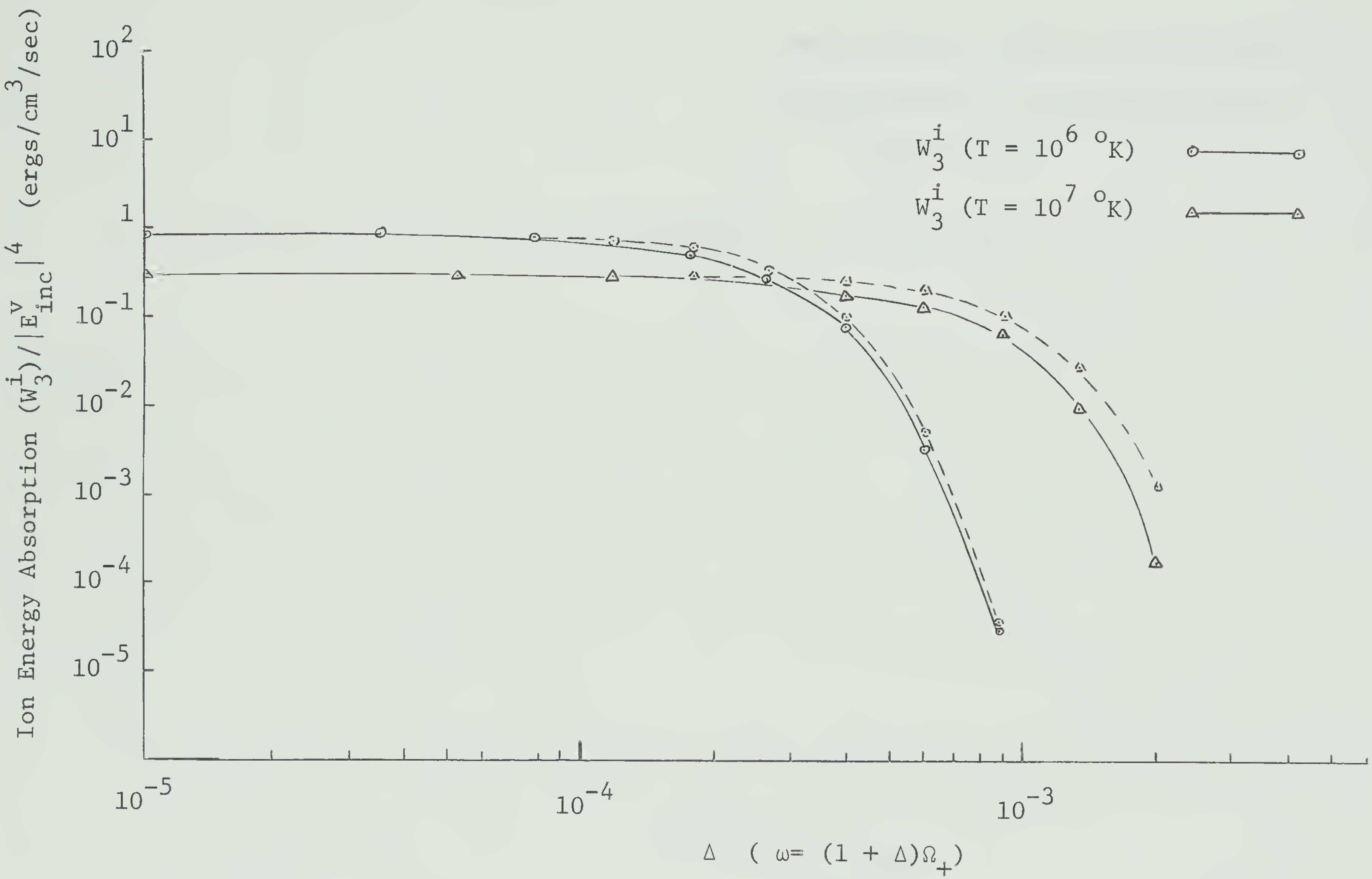

Figure 3.8. Sensitivity of ion energy absorption to frequency fluctuations. The physical parameters are as given in Figure (3.2), with $\theta_1^s = 50.2^o$. The results for negative Δ are given by the broken curves.

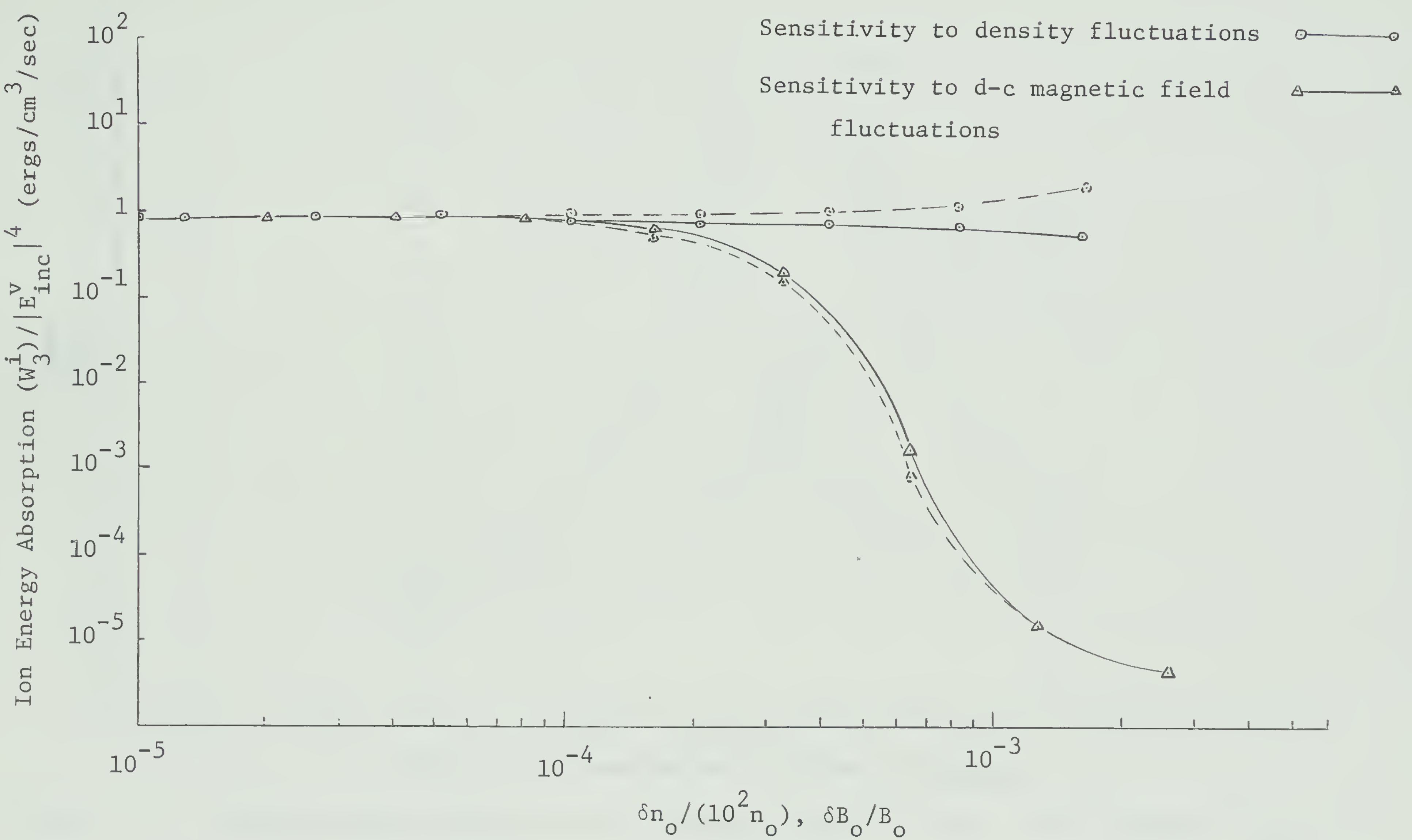

Figure 3.9. Sensitivity of ion energy absorption to density and d-c magnetic field fluctuations. The physical parameters are as given in Figure (3.2), with $\theta_1^s = 50.2^o$. The results for negative δn_o and δB_o are given by the broken curves.

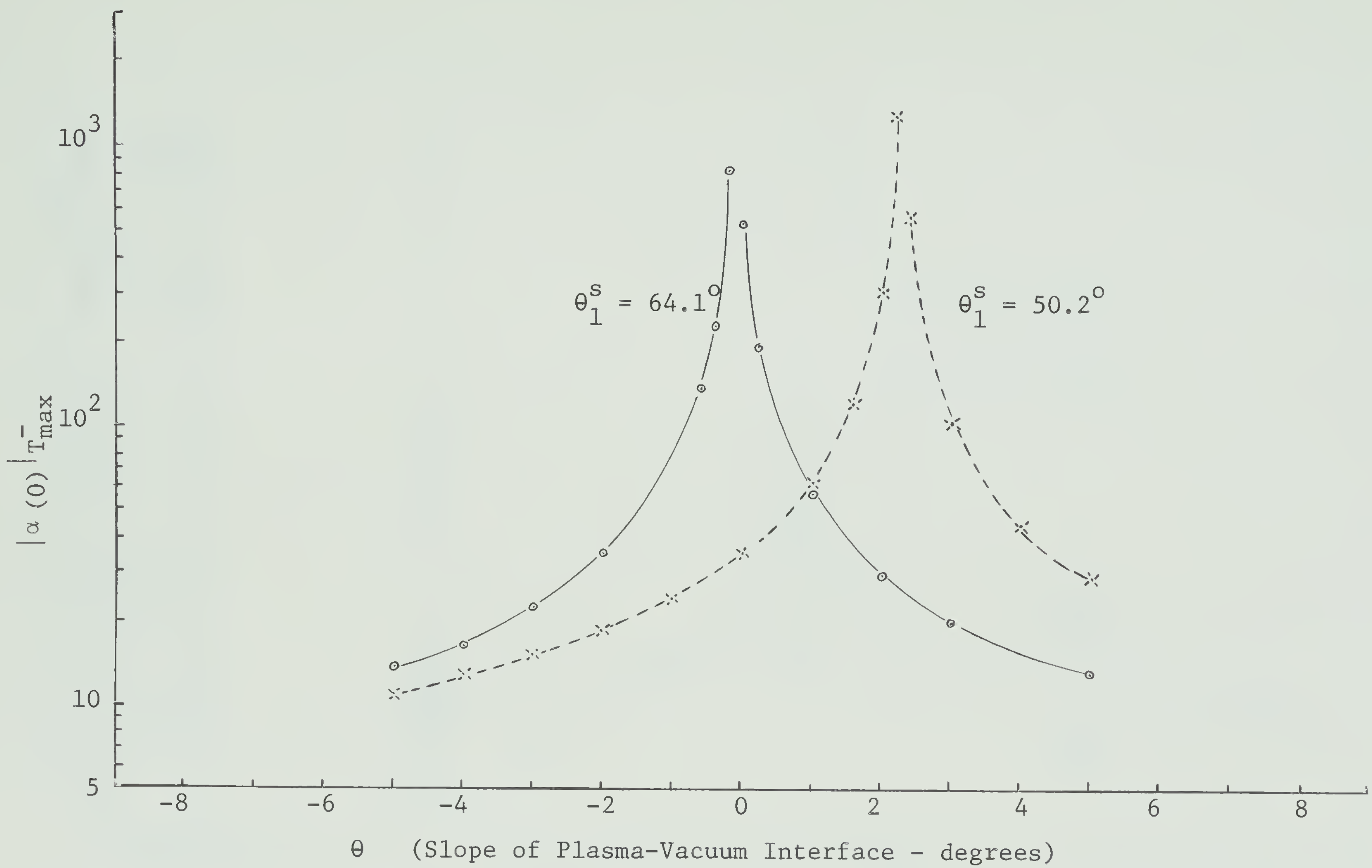

Figure 3.10. Electron Landau damping of the difference frequency wave. The parameter $\alpha^-_{(0)}$ is the ratio of the phase velocity of the difference frequency wave in the z-direction, divided by the electron thermal velocity. The physical parameters are as given in Figure (3.1), with $C_1 = 1.1$. The values of θ^s_1 are given for the condition that $\theta = 0$.

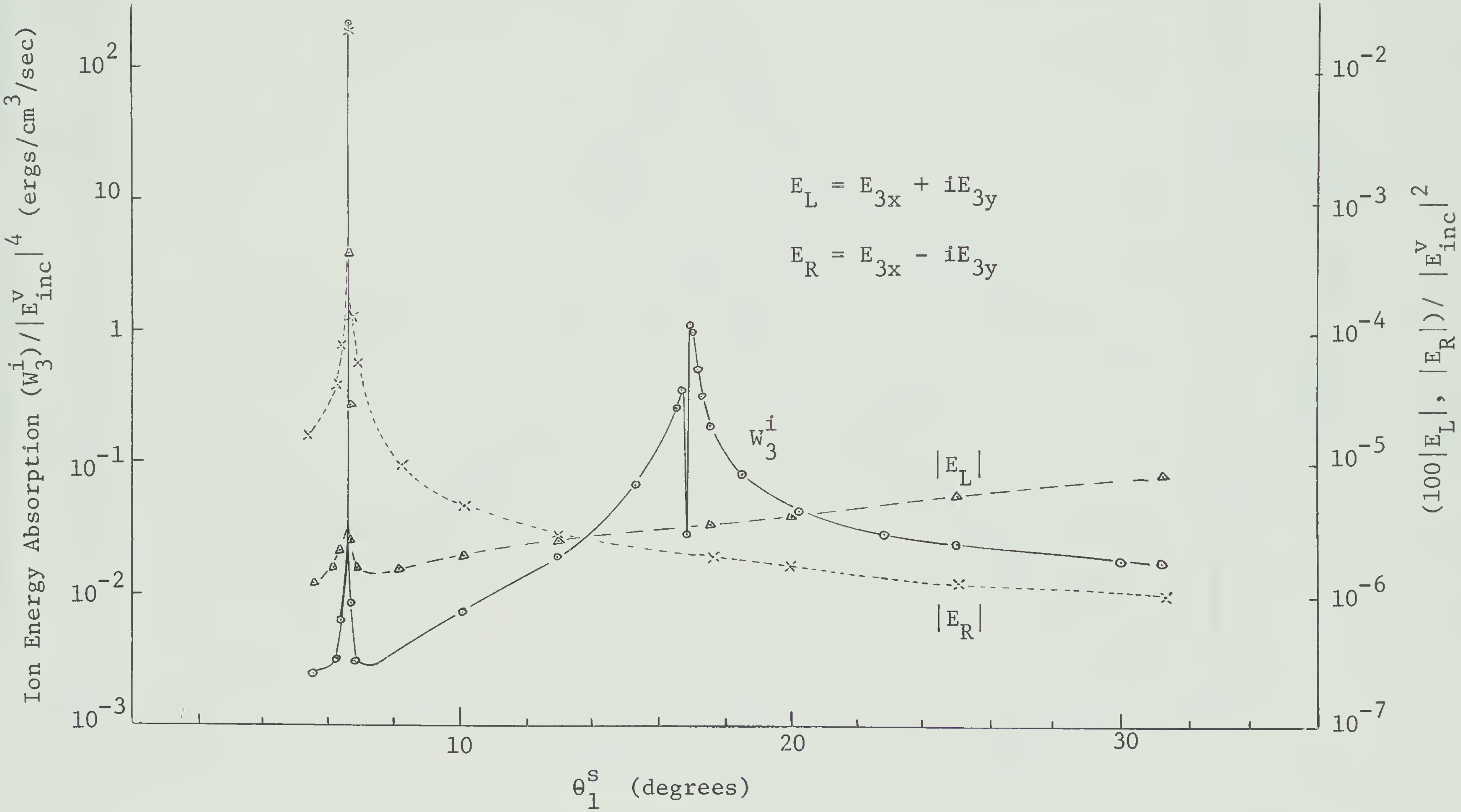

Figure 3.11. Ion energy absorption and the second order electric field as functions of the angle of incidence for the first source. The physical parameters are as given in Figure (3.1), with $C_1 = 10$, $B_o = 2.8$ Kgauss, and $\omega_1 = 1.85 \times 10^9$ Hertz.

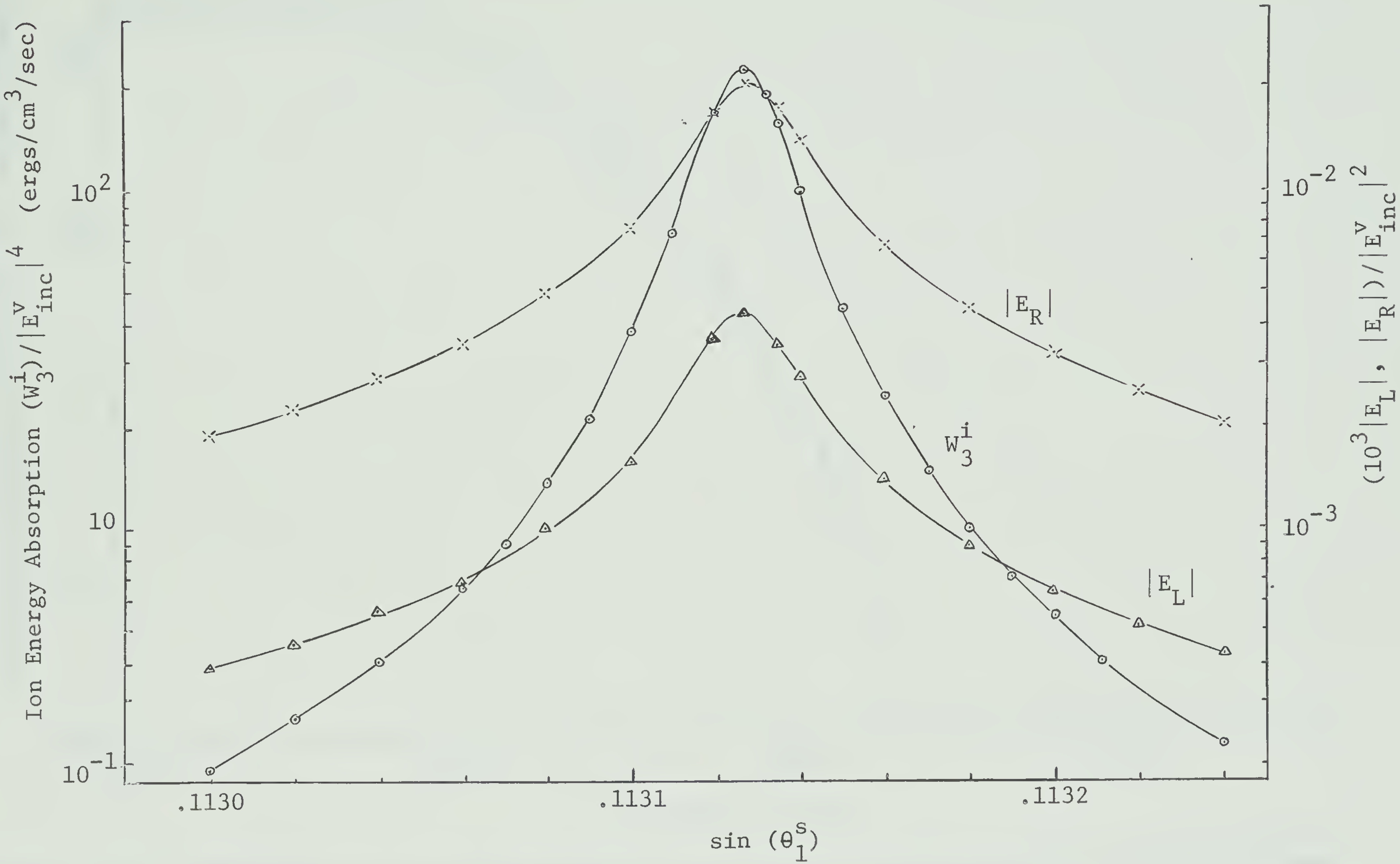

Figure 3.12. Field resonance of difference frequency wave as a function of the angle of incidence for the first source. The physical parameters are as given in Figure (3.11).

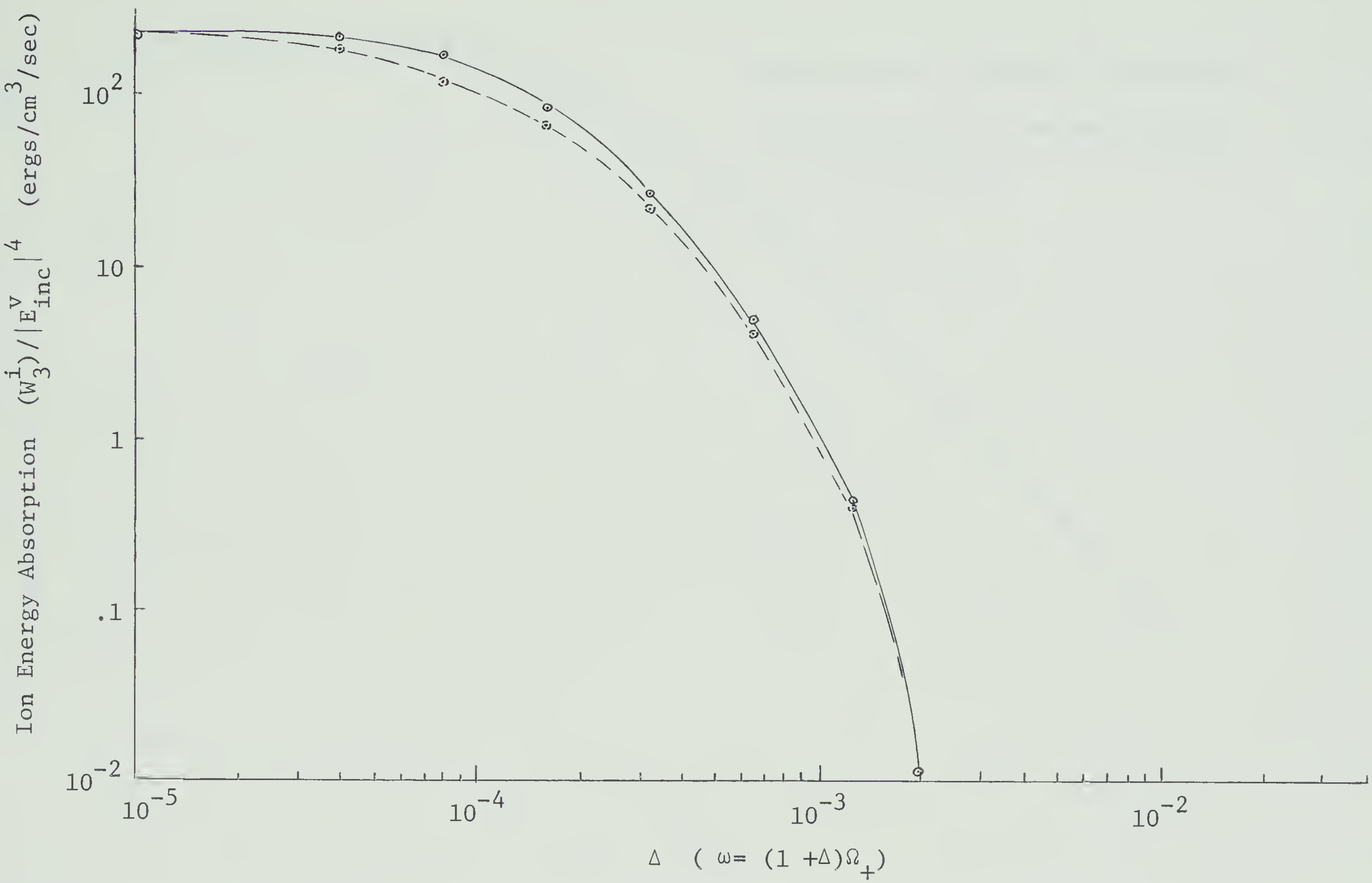

Figure 3.13. Sensitivity of ion energy absorption to frequency fluctuations at a field resonance point. The physical parameters are as in Figure (3.12), with $\sin(\theta_1^s) = 0.1131274$. The results for negative Δ are given by the broken curve.

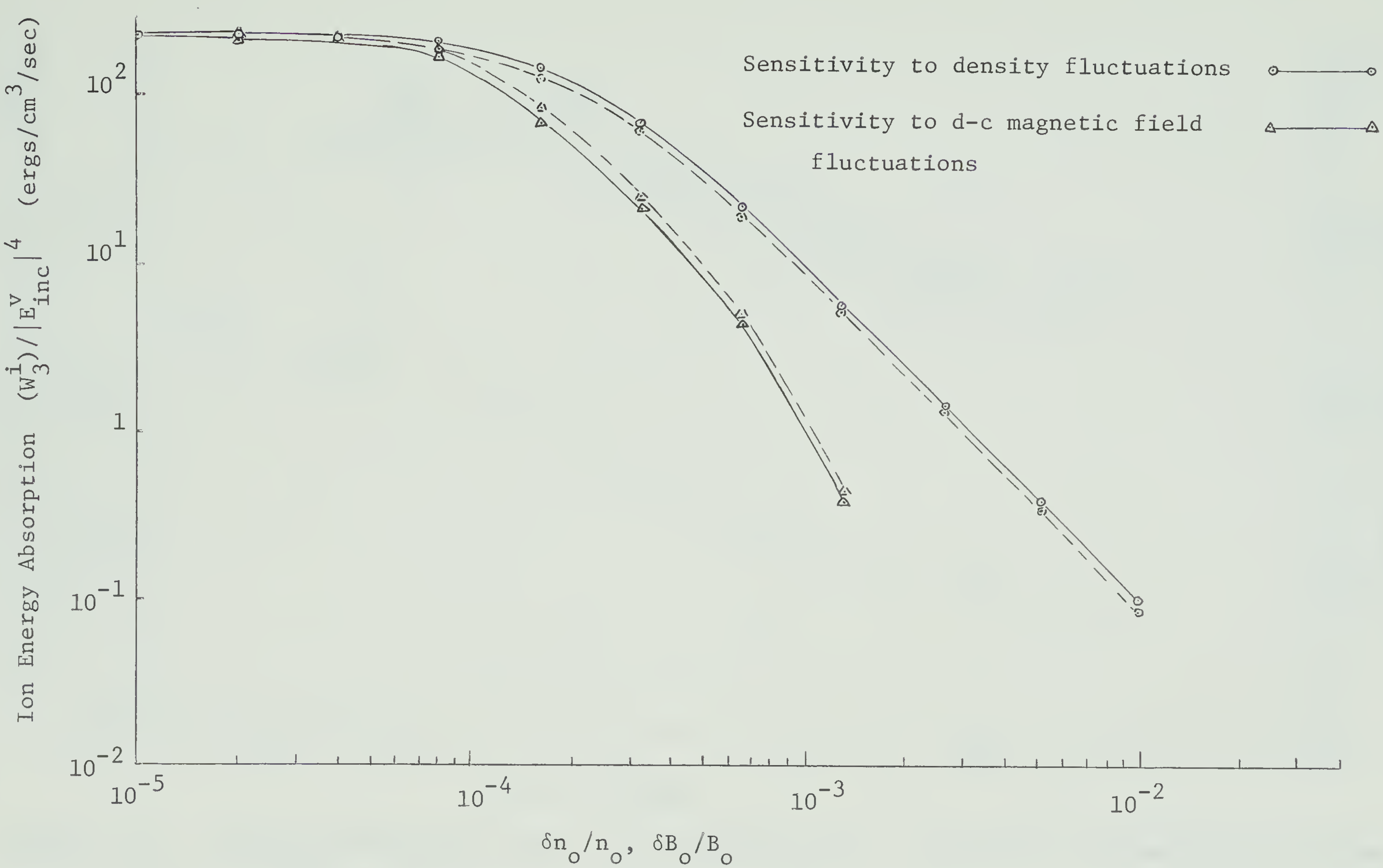

Figure 3.14. Sensitivity of ion energy absorption to density and d-c magnetic field fluctuations. The physical parameters are as in Figure (3.12), with $\sin(\theta_1^S) = 0.1131274$. The results for negative δn_o and δB_o are given by the broken curves.

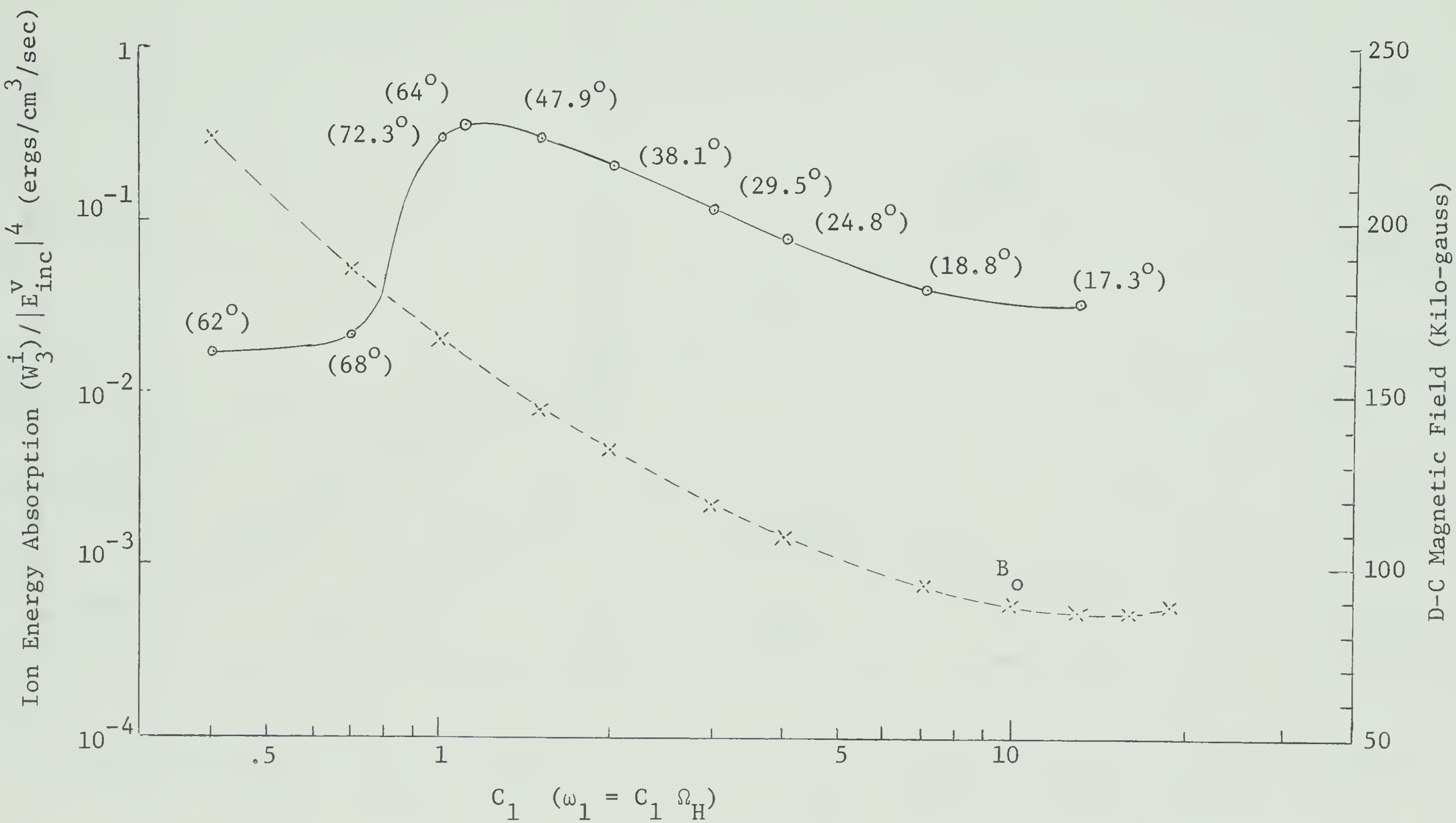

Figure 3.15. Ion Energy Absorption and B_o as a Function of C_1. The physical parameters are: plasma density = $10^{15}/cm^3$, temperature = 10^6 °Kelvin, $|E_{inc}^v| \leq 15$ KV/cm, $\lambda_{max}^+ \leq 0.5$ for $T_{max}^+ = 10^8$ °K, plasma confined for 10^{-3}sec., and $(\omega - \Omega_+)/\Omega_+ = 10^{-5}$. The d-c magnetic field is selected per equation (3.13).

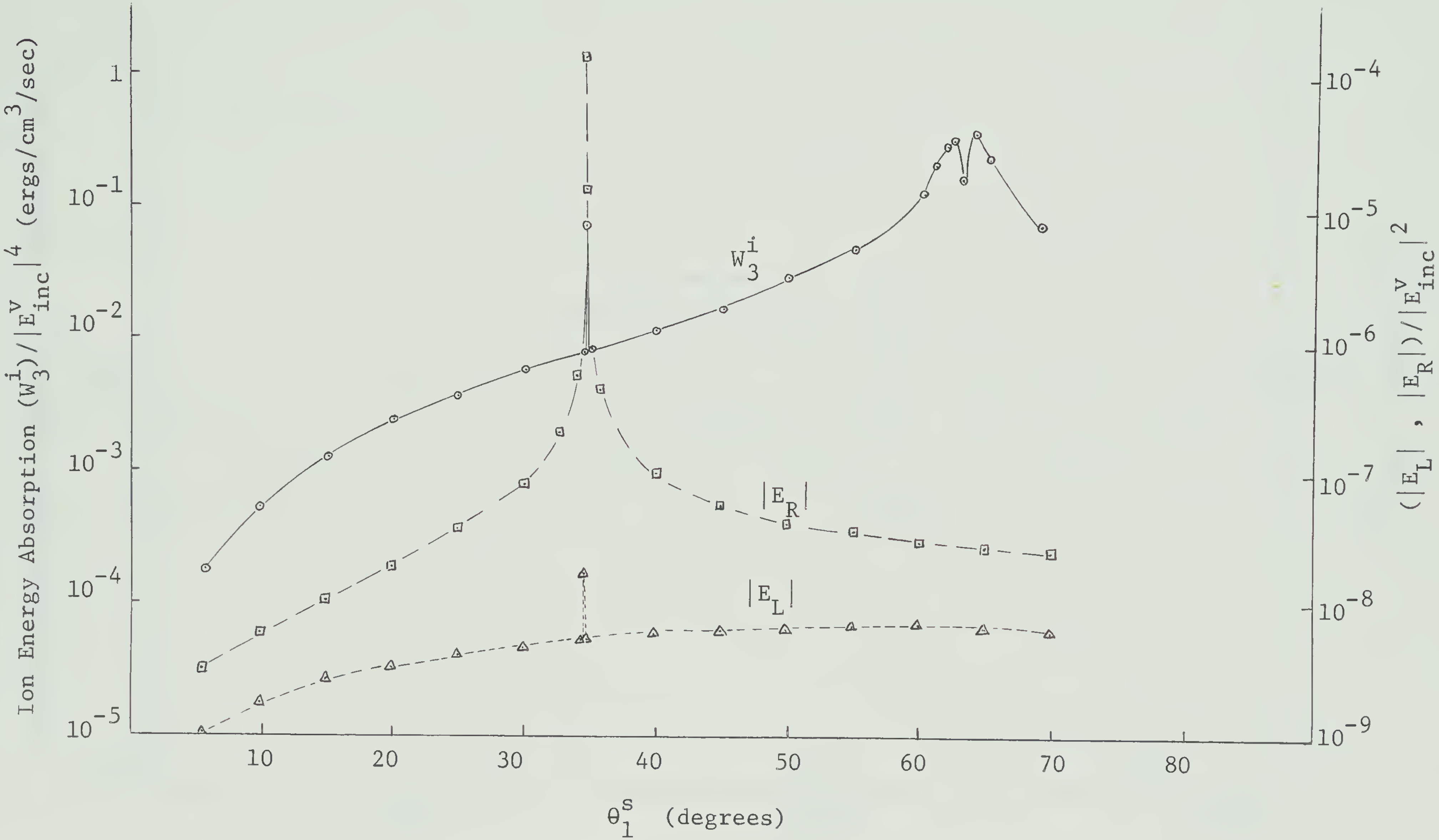

Figure 3.16. Ion energy absorption and the second order electric field as functions of the angle of incidence for the first source. The physical parameters are as given in Figure (3.15), with $C_1 = 1.1$, $B_o = 162$ Kgauss, and $\omega_1 = 1.16 \times 10^{10}$ Hertz. $E_L = E_{3x} + iE_{3y}$, $E_R = E_{3x} - iE_{3y}$

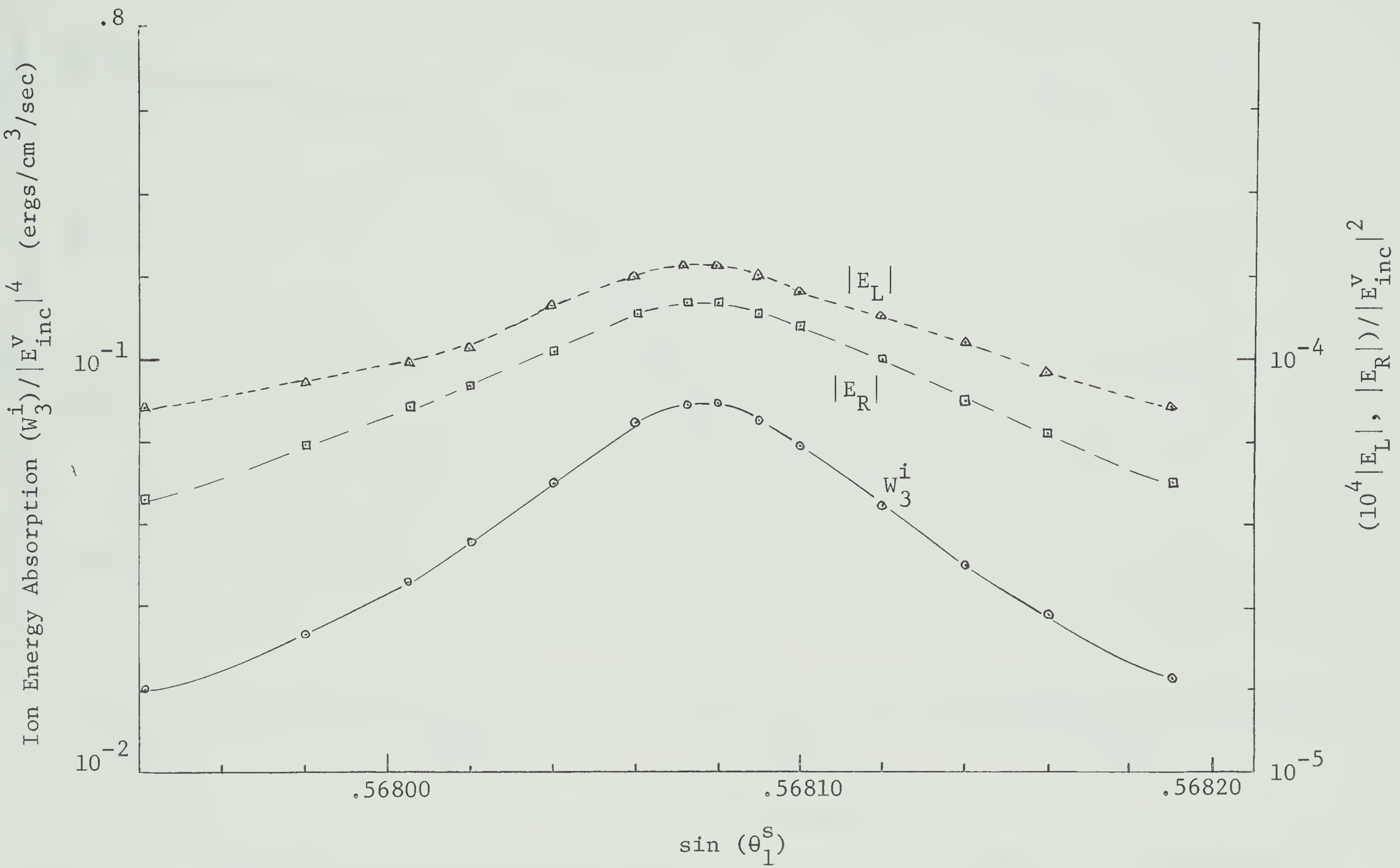

Figure 3.17. Field resonance of difference frequency wave as a function of the angle of incidence for the first source. The physical parameters are as given in Figure (3.16).

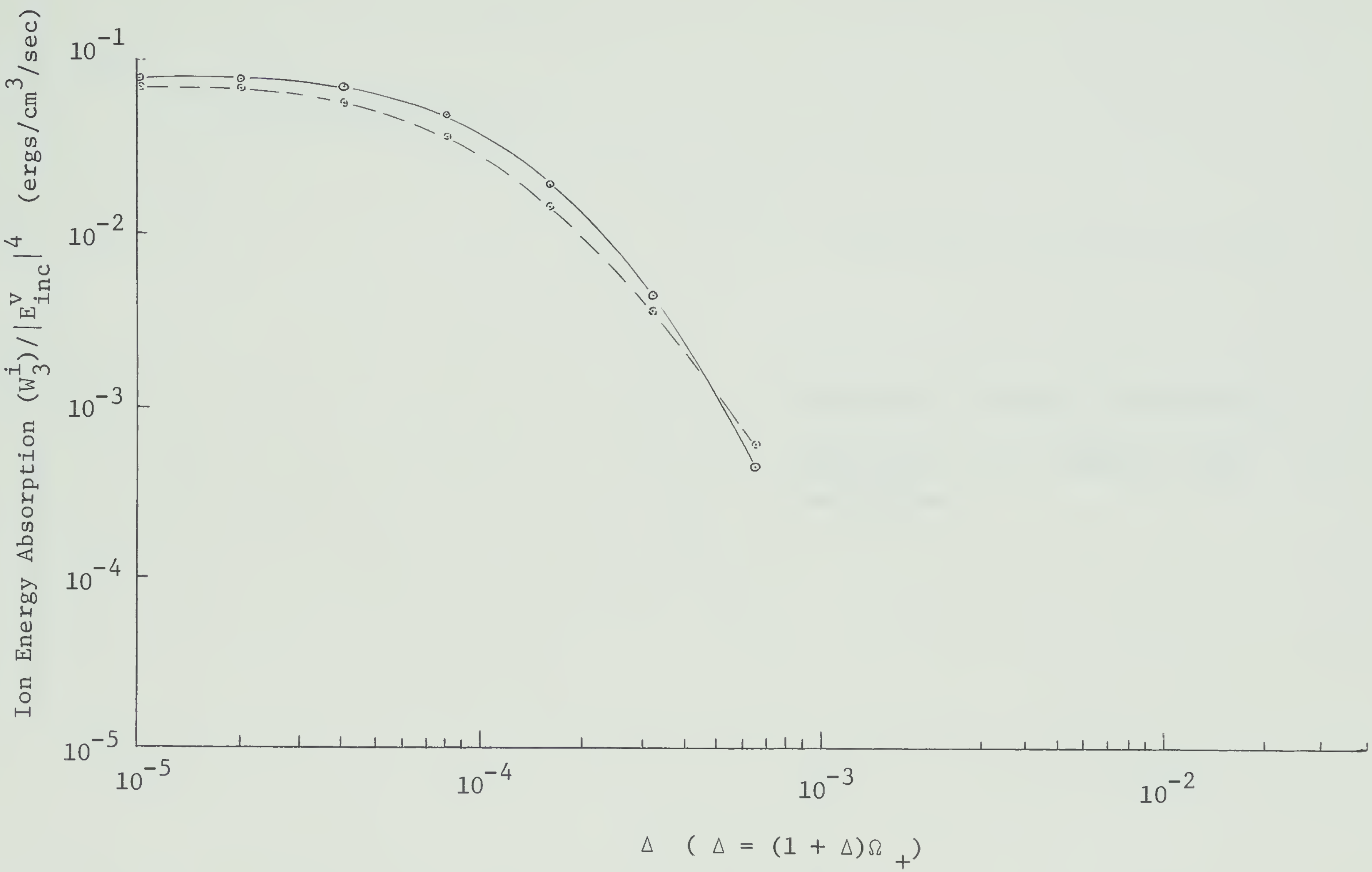

Figure 3.18. Sensitivity of ion energy absorption to frequency fluctuations at a field resonance point. The physical parameters are as in Figure (3.17), with $\sin(\theta_1^s) = 0.5680725$. The results for negative values for Δ are given by the broken curve.

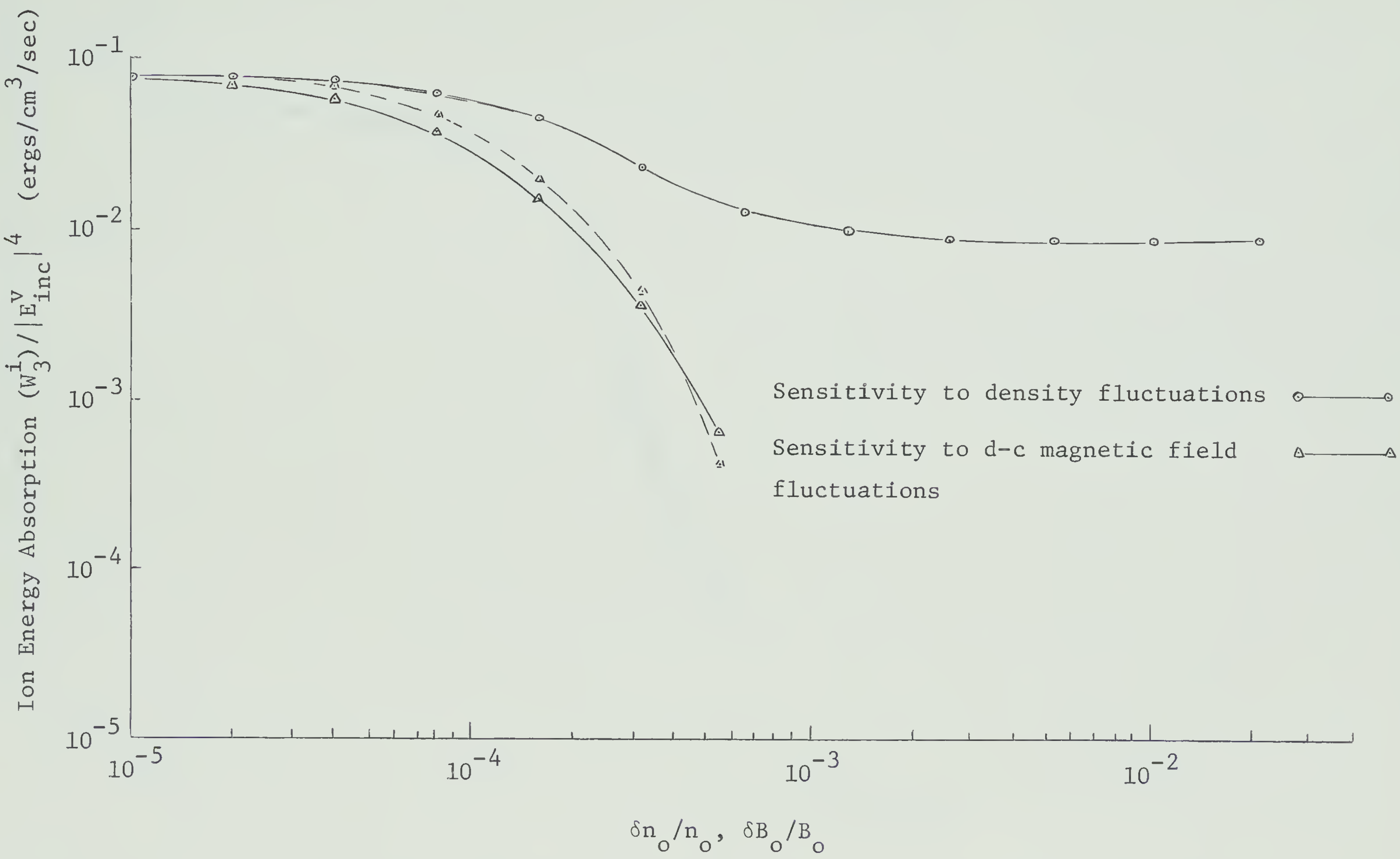

Figure 3.19. Sensitivity of ion energy absorption to density and d-c magnetic field fluctuations. The physical parameters are as in Figure (3.17), with $\sin(\theta_1^s) = 0.5680725$. The results for negative δn_o and δB_o are given by the broken curves.

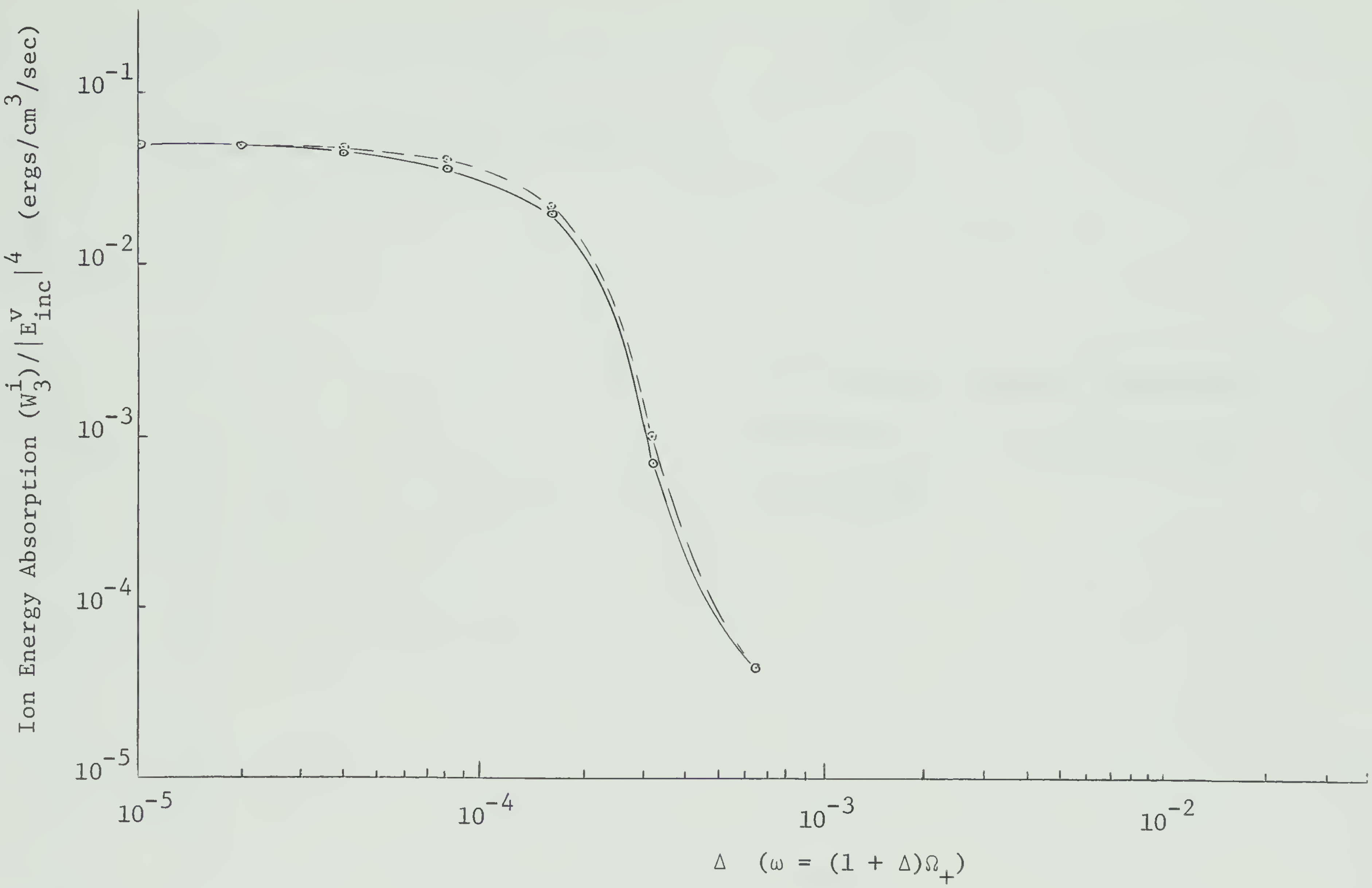

Figure 3.20. Sensitivity of ion energy absorption to frequency fluctuations. The physical parameters are as given in Figure (3.16), with $\theta_1^s = 55$ degrees. The results for negative Δ are given by the broken curve.

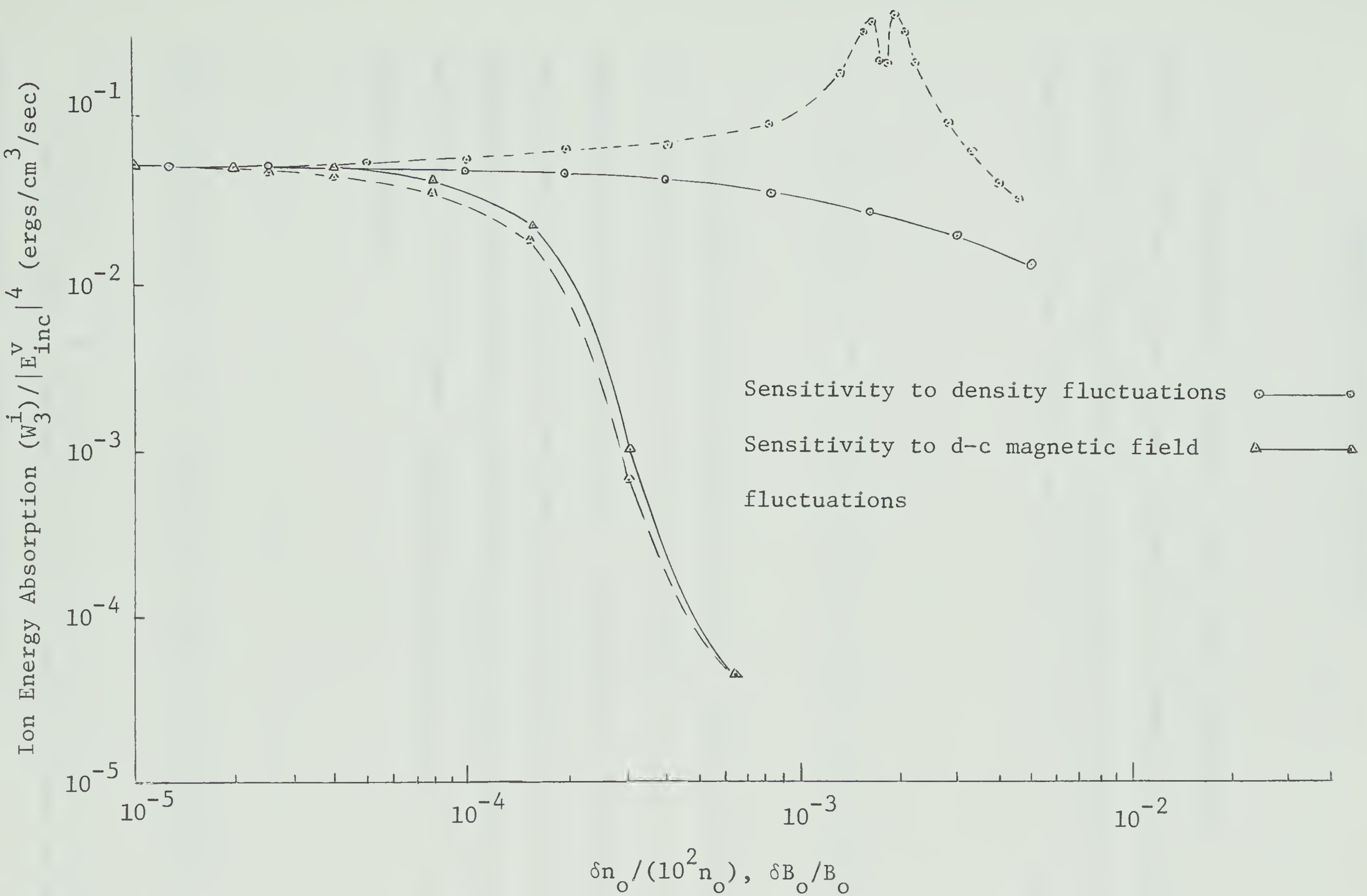

Figure 3.21. Sensitivity of ion energy absorption to density and d-c magnetic field fluctuations. The physical parameters are as given in Figure (3.16), with $\theta_1^s = 55$ degrees. The results for negative δn_o and δB_o are given by the broken curves.

CHAPTER 4 SUMMARY AND DISCUSSION OF HEATING TECHNIQUES USING THE NONLINEAR MIXING OF TWO WAVES

In this chapter, a summary of the results pertaining to ion heating through the non-linear mixing of two waves will be given. The merits of the techniques that may be used to optimize the energy absorbed by the ions from the second order fields will be discussed.

4.1 HF Heating of Ions by Maximizing the Second Order Fields

Two schemes have been suggested recently (James and Thompson [17], and Jayasimha [18]) where the collisional heating of ions is optimized by maximizing the second order fields. A resonance in the magnitude of the second order fields is obtained by allowing the mixed wave to approach a natural mode in the plasma, which in the above case is an extra-ordinary Alfven wave propagating perpendicularly to the static magnetic field.

4.1.1 HF Heating of Ions (Analysis by James and Thompson [17])

The field vector diagram for the ion heating scheme suggested by James and Thompson [17] is given in Figure (4.1).

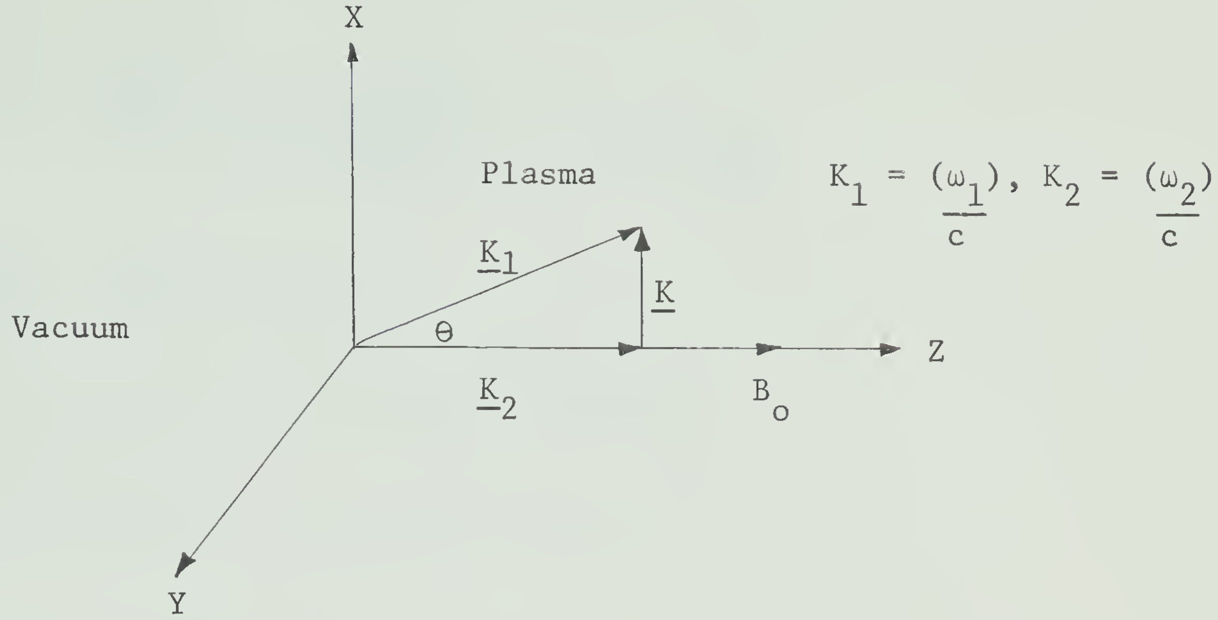

FIGURE 4.1. Field Vector Diagram for HF Ion Heating. This is essentially Figure (1) in James and Thompson[17], however the exponential dependence of the field quantities is written as to agree with the notation used in equation (2.3). The plasma is assumed to occupy the half space $z > 0$.

The authors used a cold plasma model to describe the nonlinear generation of the second order fields and the resulting ion heating. An investigation into the sensitivity of the heating process to fluctuations in frequency, density, and d-c magnetic field was carried out. These results will be compared with those obtained by the use of Whistlers in Table (4.1). The sensitivity of the heating scheme suggested by James and Thompson[17] to fluctuations in the angle of incidence for the first wave (θ in Figure (4.1)) will now be calculated. The propagation constant for an extra-ordinary Alfven wave propagating perpendicularly to the static magnetic field is

$$K^2 = \left(\frac{\omega}{c}\right)^2 \frac{(\omega_p^2 + 2\Omega_+\Omega_-)}{\left(1 + \frac{\Omega_+\Omega_-}{\omega_p^2}\right)\Omega_+\Omega_- - \omega^2\left(1 - \frac{\Omega_-^2}{\omega_p^2}\right)}$$

where $\omega_p^2 = \omega_{p-}^2(1 + m_-/m_+)$

$\omega = \omega_1 - \omega_2$ = difference frequency

In the limit that $\omega^2 << \Omega_+\Omega_-$ and $\Omega_+ << \omega_p$ [17]

$$K^2 \simeq \left(\frac{\omega}{c}\right)^2 \left(\frac{\omega_{p+}}{\Omega_+}\right)^2 \qquad \text{...(4.1)}$$

Equation (4.1) and the requirement that the mixed wave propagate perpendicularly to the static magnetic field may be used to give the required value for the difference frequency as[17]

$$\omega \sim 2\frac{(\Omega_+\Omega_-)}{\omega_{p-}^2}\omega_1 \qquad \text{...(4.2)}$$

in the limit that $\omega^2 << \Omega_+\Omega_-$, $\omega_1 << \sqrt{\frac{m_+}{m_-}}\,\omega_p$ & $\omega^2 << \omega_{p+}^2$

If $\theta_\perp$ is the required angle of incidence for the first wave so that $\underline{K}$ is perpendicular to the static magnetic field, the use of the free space values for the propagation vectors $\underline{K}_1$ and $\underline{K}_2$ in Figure (4.1) yields

$$\sin^2\theta_\perp = \omega(\omega_1 + \omega_2)/\omega_1^2 \qquad \text{...(4.3)}$$

If the error in setting the required value of θ is θ_ε, then for small values of θ_ε, the x-component of the propagation vector for the mixed wave is

$$K_x = \omega_1/c \ \sin(\theta_\perp + \theta_\varepsilon)$$

$$= \omega_1/c \ \sin\theta_\perp \left(\cos\theta_\varepsilon + \cos\theta_\perp \sin\theta_\varepsilon / \sin\theta_\perp\right)$$

but $(\omega_1/c)\sin\theta_\perp = K_{\text{Alfvén}}$ = required value for K for a field resonance. For θ_ε small,

$$K_x \sim K_{\text{Alfvén}} \left(1 + \theta_\varepsilon / \tan\theta_\perp\right) \qquad \ldots(4.4)$$

No resonance effects will be observed when

$$\theta_\varepsilon / \tan\theta_\perp \sim O(1)$$

Equations (4.2) and (4.3) may be used to show that the above condition is equivalent to

$$\theta_\varepsilon \sim 2\sqrt{\Omega_+\Omega_-} / \omega_{P-} \qquad \ldots(4.5)$$

If the sensitivity of the heating process to angular fluctuations is defined by the quarter-power points, the analysis given by James and Thompson[17] may be extended to give

$$\theta_\varepsilon < \omega\delta / (\Omega_H \omega_{P-}) = \frac{2\Omega_H \omega_1 \delta}{\omega_{P-}^3} \qquad \ldots(4.6)$$

where $\delta = \delta_{ei}(1 + m_-/m_+)$

Condition (4.6) is much more severe than is (4.5). For the example considered in [17], (4.6) gives

$$\theta_\varepsilon \ll 10^{-8} \text{ radians} \qquad \ldots(4.7)$$

For the same case, if any resonance effect is to be observed at all, equation (4.5) gives the requirement that

$$\theta_\varepsilon \ll .05 \text{ radians} \quad (\sim 3 \text{ degrees}) \qquad \ldots(4.8)$$

4.1.2 HF Heating of Ions (Analysis by Jayasimha[18])

In this analysis, two incident waves with frequencies approximately equal to the electron plasma frequency are sent into the plasma at right angles to the static magnetic field. The geometry of the plasma and the sources is given in Figure (4.2)

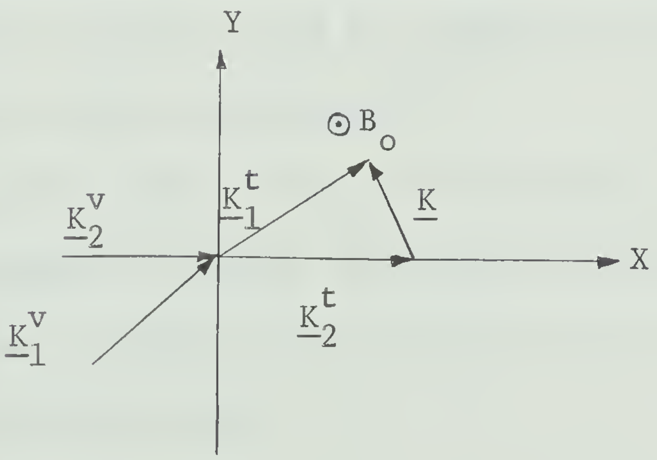

Figure 4.2. Field Vector Diagram for HF ion Heating used by Jayasimha[18].

The results obtained for this case are closely related to those obtained by James and Thompson[17]. By the use of lower frequency incident waves, a gain of approximately four hundred is claimed in the power absorbed by the ions from the mixed wave. An improvement in the allowable fluctuations in the angular position of the sources is also realized through the use of lower frequency waves. However, this gain in the rate of energy absorption by the ions may not be realized in practice because of the tighter restrictions on frequency, density, and d-c magnetic field fluctuations. The sensitivity of the heating process to frequency, density and d-c magnetic field fluctuations are summarized below for a plasma with a density of $10^{12}/\text{cm}^3$ and a temperature of 10^6 degrees Kelvin.

$$\begin{aligned} \frac{\delta\omega_2}{\omega_2} &\sim \frac{m_-}{m_+} N^2 \frac{\Delta_{ei}}{\omega_{p-}} \sim 3.4\times 10^{-10} \\ \frac{\delta n_0}{n_0} &\sim 2N \frac{\Delta_{ei}}{\Omega_+} \sim 10^{-6} \\ \frac{\delta B_0}{B_0} &\sim N \frac{\Delta_{ei}}{\Omega_+} \sim 10^{-6} \end{aligned} \qquad \ldots(4.9)$$

where $N = \omega/\Omega_+$. The above results are an order of magnitude more sensitive to frequency, density, and d-c magnetic field fluctuations than those obtained by James and Thompson[17].

The finite Larmor radius effects would appear to play a role more important than is suggested because the dominant electron driving terms cancel, leaving terms of order (m_-/m_+), and this is of the same order as the dominant ion driving terms.

4.2 Comparison Between HF and Whistler Heating

The use of high frequency waves ($\omega_1, \omega_2 > \Omega_-, \omega_{p-}$) as suggested by James and Thompson[17] is compared to the use of Whistler waves ($\Omega_+ < \omega_1 < \Omega_-$) as suggested in this thesis, in Table (4.1). A comparison is made between the results obtained for heating plasmas with a density of $10^{12}/cm^3$ and a temperature of 10^6 °Kelvin. Two possible modes of operation for Whistler heating are presented. One is the use of the resonance obtained in the second order ion current by allowing the difference frequency to approach the ion cyclotron frequency. The physical parameters ($n_o, B_o, \omega_1, \theta_1^s, \ldots$) are chosen to be those given in Figures (3.6) and (3.8). The second method uses an ion current resonance coupled with the field resonance that is obtained by having the difference frequency wave approach a natural mode in the plasma. The physical parameters are chosen to be those given in Figure (3.12). The claims suggested by Jayasimha[18] have already been compared to the results given by James and Thompson[17] in Section (4.1.2) and will not be repeated here.

The results obtained for the cases where a second order current and field resonance are used to optimize the power absorbed by the ions (eg. see Figure (3.11)) will be valid for incident electric field amplitudes less than 3 KV/cm. The field resonance effect will cause the second order fields to approach the same order of magnitude as the incident fields for larger values of $|E^v_{inc}|$, giving rise to secular terms. This may be resolved by considering the mixed fields to be of the same order as the incident fields in the perturbation

$n_o = 10^{12}/cm^3$ $T = 10^6$ °K	HF HEATING JAMES & THOMPSON[17]	WHISTLER HEATING		
ION PWR. ABSORPTION OPTIMIZED BY	FIELD RESONANCE	SECOND ORDER ION CURRENT RESONANCE		ION CURRENT + FIELD RESONANCE
PHYSICAL PARAMETERS	AS IN REFERENCE [4]	AS IN FIGURE (3.6)	AS IN FIGURE (3.8)	AS IN FIGURE (3.12)
INCIDENT FREQ. (ω_1) (HERTZ)	3.5×10^{10}	3.7×10^8	3.7×10^8	1.85×10^9
STATIC MAGNETIC FIELD (B_o) K gauss	3.2	5.2	5.2	2.8
ANGLE OF INCIDENCE θ_1^s (DEGREES)	5.35	64.1	50.2	6.5
POWER REFLECTED AT PLAS-VAC. INTERFACE	NEGLIGIBLE	30%	32%	23%
$\frac{\delta\omega_1}{\omega_1}$	10^{-8}	$.9 \times 10^{-6}$	1.2×10^{-5}	$.91 \times 10^{-6}$
$\frac{\delta n_o}{n_o}$	10^{-5}	.2	>.2	$.74 \times 10^{-3}$
$\frac{\delta B_o}{B_o}$	$.5 \times 10^{-5}$	$.42 \times 10^{-4}$	$.56 \times 10^{-3}$	$.39 \times 10^{-3}$
$\delta(\theta_1^s)$ (DEGREES)	$<.57 \times 10^{-6}$	<5	<30°	$<.9 \times 10^{-4}$
$(W_3^1)/\lvert E_{inc}^v\rvert^4$ ergs/sec/cm^3	.65	13	.94	2.3×10^2

TABLE 4.1 COMPARISON BETWEEN HF AND WHISTLER HEATING.

theory[30,31]. The perturbation theory used in this thesis is valid for field intensities to 70 KV/cm in the cases where the energy absorbed by the ions is optimized only through an ion current resonance.

4.3 Conclusions

From Table (4.1) it is evident that the sensitivity of a heating process employing the nonlinear mixing of whistlers to frequency, density, d-c magnetic field and angular fluctuations is considerably less than that in which HF waves are used. If only an ion current resonance effect is utilized, the significant gain is in the relaxation of the stringent restrictions on density and angular fluctuations. Through the use of whistlers, the sensitivity to density fluctuations is reduced by four orders of magnitude while the sensitivity to angular fluctuations is reduced by about seven orders of magnitude. An order of magnitude is gained in the rate of ion energy absorption. The sensitivities to frequency and d-c magnetic field fluctuations are reduced by two and one order of magnitude respectively. Since the use of an ion current resonance effect only requires that the difference frequency be sufficiently close to the ion cyclotron frequency, the only critical adjustments in an experiment will be that of the difference frequency and the d-c magnetic field.

If a field resonance effect is utilized, in addition to an ion current resonance the mixed wave must approach a natural mode in the plasma, which in the cases considered, is an extraordinary Alfven wave. This results in much tighter restrictions on frequency, density, d-c magnetic field, and angular fluctuations than is required in the case where only an ion current resonance effect is utilized.

Two advantages of the use of HF waves, rather than whistlers in the mixing process are:

(i) No power is reflected at the plasma-vacuum interface when HF waves are used.

(ii) The transmission of HF waves into a plasma will be independent of the plasma-vacuum interface geometry.

In the analysis performed by James and Thompson[17], it was found that $E_{3y} = i\Omega_+/\omega\ E_{3x}$. For the case considered in [17], $\Omega_+/\omega \sim 1/8$. That is, the second order electric field was right hand elliptically polarized. If ω were allowed to approach Ω_+, this would suggest that the second order electric field would become almost RH circularly polarized. This is just what is found in the case of whistler heating near field resonance with ω near Ω_+ (See Figure (3.2). For both the ion cyclotron and collisional damping of the difference frequency wave, the ions resonate only with the electric field component rotating in the direction of ion gyration in the static magnetic field. This would then suggest that a possible modification of the heating scheme presented in this thesis would be to have the ions absorb energy from the difference frequency wave near the second harmonic of the ion cyclotron frequency so as to increase the amount of LH

polarization. This aspect would certainly warrant further investigation.

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APPENDIX A. METHOD OF ORBIT INTEGRATIONS APPLIED TO SECOND ORDER

The Vlasov equation is

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial r} + \frac{Z_j e \varepsilon_j}{m_j} \left(\underline{E} + \frac{\underline{v} \times \underline{B}}{c} \right) \cdot \frac{\partial f}{\partial \underline{v}} = 0 \qquad \text{...(A.1)}$$

Equation (A.1) will be solved assuming the following ordering of variables;

$$\begin{bmatrix} f \\ \underline{E} \\ \underline{B} \end{bmatrix} = \begin{bmatrix} f_0 \\ 0 \\ \underline{B}_0 \end{bmatrix} + \begin{bmatrix} f_1 \\ \underline{E}_1 \\ \underline{B}_1 \end{bmatrix} e^{i(\underline{K}_1 \cdot \underline{r} - \omega_1 t)} + \begin{bmatrix} f_2 \\ \underline{E}_2 \\ \underline{B}_2 \end{bmatrix} e^{i(\underline{K}_2 \cdot \underline{r} - \omega_2 t)}$$

$$+ \begin{bmatrix} f_3 \\ \underline{E}_3 \\ \underline{B}_3 \end{bmatrix} e^{i(\underline{K} \cdot \underline{r} - \omega t)} + \text{Complex Conjugate} + \text{Higher Order Terms}$$

The perturbed distribution functions will be solved for in a Lagrangian system of coordinated. The change of a distribution function expressed along some trajectory defined by $\underline{r}(t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \underline{r}} \cdot \frac{d\underline{r}}{dt} + \frac{\partial f}{\partial \underline{v}} \cdot \frac{d\underline{v}}{dt} \qquad \text{where} \qquad \underline{v} = \frac{d\underline{r}}{dt}$$

In this particular case, the zero'th order trajectory of a particle in a magnetic field will be chosen, that is

$$\frac{d\underline{r}}{dt} = \underline{v} \qquad \frac{d\underline{v}}{dt} = \frac{Z_j e \varepsilon_j}{m_j} \frac{\underline{v} \times \underline{B}_0}{c} \qquad \varepsilon_j = \text{sign of species}$$

Substituting (A.2) into (A.1)

$$\frac{\partial}{\partial t}\left(f_0 + f_1 e^{i(\underline{K}_1\cdot\underline{r}-\omega_1 t)} + f_2 e^{i(\underline{K}_2\cdot\underline{r}-\omega_2 t)} + f_3 e^{i(\underline{K}\cdot\underline{r}-\omega t)} + \text{Complex Conjg.}\right)$$

$$+ \underline{v}\cdot\frac{\partial}{\partial \underline{r}}\left(f_0 + f_1 e^{i(\underline{K}_1\cdot\underline{r}-\omega_1 t)} + f_2 e^{i(\underline{K}_2\cdot\underline{r}-\omega_2 t)} + f_3 e^{i(\underline{K}\cdot\underline{r}-\omega t)} + \text{Complex Conjg.}\right)$$

$$+ \frac{Z_j e \mathcal{E}_j}{m_j}\left\{\left[\underline{E}_1 e^{i(\underline{K}_1\cdot\underline{r}-\omega_1 t)} + \underline{E}_2 e^{i(\underline{K}_2\cdot\underline{r}-\omega_2 t)} + E_3 e^{i(\underline{K}\cdot\underline{r}-\omega t)} + \text{Complex Conjg.}\right]\right.$$

$$\left. + \frac{\underline{v}\times}{c}\left[\underline{B}_0 + \underline{B}_1 e^{i(\underline{K}_1\cdot\underline{r}-\omega_1 t)} + \underline{B}_2 e^{i(\underline{K}_2\cdot\underline{r}-\omega_2 t)} + \underline{B}_3 e^{i(\underline{K}\cdot\underline{r}-\omega t)} + \text{Complex Conjg.}\right]\right\}$$

$$\cdot \frac{\partial}{\partial \underline{v}}\left\{ f_0 + f_1 e^{i(\underline{K}\cdot\underline{r}-\omega_1 t)} + f_2 e^{i(K_2\cdot\underline{r}-\omega_2 t)} + f_3 e^{i(\underline{K}\cdot\underline{r}-\omega t)} + \text{Complex Conjg.}\right\}$$

$$= 0$$

The equations for the various perturbations are now obtained by using harmonic balance. For instance, by equating terms with exponential dependence $i(\underline{K}\cdot r - \omega t)$.

$$\frac{\partial f_3}{\partial t} + \underline{v}\cdot\frac{\partial f_3}{\partial r} + \frac{Z_j e \mathcal{E}_j}{m_j}\left(\frac{\underline{v}\times\underline{B}_0}{c}\right)\cdot\frac{\partial f_3}{\partial v} = -\frac{Z_j e \mathcal{E}_j}{m_j}\left(\underline{E}_3 + \frac{\underline{v}\times\underline{B}_3}{c}\right)\cdot\frac{\partial f_0}{\partial v}$$

$$-\frac{Z_j e \mathcal{E}_j}{m_j}\left(\underline{E}_1 + \frac{\underline{v}\times\underline{B}_1}{c}\right)\cdot\frac{\partial f_2^*}{\partial v} - \frac{Z_j e \mathcal{E}_j}{m_j}\left(\underline{E}_2^* + \frac{\underline{v}\times\underline{B}_0}{c}\right)\cdot\frac{\partial f_1}{\partial v} \qquad \ldots(A.3)$$

The left hand side of equation (A.3) is just the change of the distribution function f_3 along the zero'th order trajectory of a particle in a magnetic field. The solution of equation (A.3) and its velocity moments will give the second order current. This may then be used with Maxwell's equations to give the second order fields and the resulting dissipation can then be found.

APPENDIX B METHOD OF SOLUTION FOR f_{32}

The asymptotic form of the assumed solutions for the incident waves is such that as $t \to -\infty$, the value of the electric and magnetic field components tend to zero. From equation (2.3), this implies that $Im(\omega_1) > 0$. The imaginary component of ω_1 will be denoted as ω_{1i}. A similar argument applies to the second incident wave, so that the imaginary component of the mixed frequency may be written as

$$\omega_i = Im(\omega_1) + Im(\omega_2) \qquad \text{...(B.1)}$$

From equation (2.10)

$$f_1(t') = \lim_{T_1 \to \infty} \int_{-T_1}^{t'} dt'' \; (\quad) \; E_{1x} \, e^{\omega_{1i} t''} \qquad \text{...(B.2)}$$

where all constants, terms linearly dependent upon t'' and periodic terms as $\exp(-i\,\omega_{1r} t'')$ have been incorporated into the term denoted by (   ). The results given in equations (2.4) and (2.5) may be used to express the variables at time t'' in terms of their values at time t'. If τ' is defined as follows:

$$\tau' = t' - t''$$

equation (B.2) may be rewritten as

$$f_1(t') = \lim_{T_1 \to \infty} \int_{0}^{t'+T_1} d\tau' \; (\quad) \; E_{1x} \, e^{\omega_{1i}(t' - \tau')} \qquad \text{...(B.3)}$$

If the above expression is substituted into equation (2.49), then

$$f_{32} = \lim_{\substack{T \to \infty \\ T_1 \to \infty}} \int_{-T}^{t} dt' \,[\qquad]\, E_{1x} E_{2x}^{*} e^{\omega_i t'} \int_{0}^{t'+T_1} d\tau \,(\qquad)\, e^{-\omega_{1i}\tau'} \qquad \ldots (B.4)$$

where constant and periodic terms have been incorporated into the term denoted by [ ]. The condition that t'' $\leq$ t', as given by the integration limits in equation (B.2) requires that $T_1 \geq T$ in equation (B.4). The ambiguity in the definition of the upper limit in the second integral in equation (B.4) as $T \to \infty$ will be resolved by writing this equation as a sum of two terms, the second of which will be shown to be zero.

$$f_{32} = \lim_{\substack{T \to \infty \\ T_1 \to \infty}} \int_{-T(1-\varepsilon)}^{t} dt' \, E_{1x} E_{2x}^{*} \,[\qquad]\, e^{\omega_i t'} \int_{0}^{t'+T_1} d\tau' \,(\qquad)\, e^{-\omega_{1i}\tau'}$$

$$+ \lim_{\substack{T \to \infty \\ T_1 \to \infty}} \int_{-T}^{-T(1-\varepsilon)} dt' \, E_{1x} E_{2x}^{*} \,[\qquad]\, e^{\omega_{1i} t'} \int_{0}^{t'+T_1} d\tau' \,(\qquad)\, e^{-\omega_{1i}\tau'}$$

$$\ldots (B.5)$$

For ε small, but non-zero, and with the condition that $T_1 \geq T$, the second integration in the first term in equation (B.5) may be written as

$$\int_{0}^{\infty} d\tau' \,(\qquad)\, e^{-\omega_{1i}\tau'}$$

As $T \to \infty$, the upper limit tends to $\varepsilon\,\infty$ in the worst case, which may be replaced by ∞ if ε is small but non-zero. Consequently the first integration in the first term on the RHS in equation (B.5) defines an integration on the interval ($-\infty$,t]. The second integration in the first term may be taken over the closed interval [0, ∞]. The first

term on the RHS of equation (B.5) with $\varepsilon \to 0$ is that given by equation (2.50).

The absolute value of the second term in (B.5) will be shown to be smaller than any positive number and therefore zero as $\varepsilon \to 0$. Let

$$\Delta f_{32} = \lim_{\substack{T \to \infty \\ T_1 \to \infty}} \int_{-T}^{-T(1-\varepsilon)} dt' E_{1x} E_{2x}^{*} [\quad] e^{\omega_i t'} \int_{0}^{t'+T_1} d\tau' (\quad) e^{-\omega_{1i} \tau'} \qquad \ldots (B.6)$$

Since $\omega_{1i} > 0$, the absolute value of the second integral is bounded. This value will be denoted by M_1. The maximum value of the term, $E_{1x}E_{2x}^{*}$ [ ] will be taken to be less than some finite value M_2. Then

$$\Delta f_{32} \le \lim_{T \to \infty} \int_{0}^{\varepsilon T} d\tau M_1 M_2 e^{\omega_i(\tau - T)} = \lim_{T \to \infty} \frac{M_1 M_2}{\omega_i} e^{-\omega_i T} \left(e^{\varepsilon \omega_i T} - 1\right) \qquad \ldots (B.7)$$

By choosing ε sufficiently small, the RHS of equation (B.7) may be made less than any positive number. Therefore $\Delta f_{32} = 0$.